An Empirical Comparison among Four Estimation Methods for the Laplace Distribution and Its Potential Application in Medical Research

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Abstract: This study investigates the performance of four parameter estimation methods for the Laplace distribution: Method of Moments (MM), Maximum Likelihood Estimation (MLE), Minimum Chi-Square Estimation using equiprobable cells (MCE-EQ), and Minimum Chi-Square Estimation using Representative Points (MCE-RP). Through comprehensive Monte Carlo simulations with sample sizes ranging from 50 to 400, we compare the root mean squared error (RMSE) of the location (μ) and scale (b) parameter estimates. Our results demonstrate that while MLE remains robust for location estimation, the MCE-RP method consistently outperforms other estimators—including MLE—for the scale parameter, particularly in small to moderate samples. The use of Representative Points, which provide an optimal discretization of the distribution, significantly enhances estimation precision. These findings are especially relevant for medical research, where accurate estimation of variability—such as in biomarker concentration levels or physiological response times—is critical for reliable sample size determination, risk assessment, and clinical decision-making. MCE-RP thus offers a superior, reliable estimator for the Laplace scale parameter, with direct implications for improving statistical inference in applied biomedical studies.

Purpose: The purpose of this research is to empirically evaluate and compare the finite-sample performance of four estimation methods for the Laplace distribution's parameters, with a focus on the novel application of Representative Points in minimum chi-square estimation. This work seeks to bridge the gap between theoretical estimation methods and practical applications, providing applied researchers with a more robust estimation tool when modeling data with Laplace characteristics, such as those commonly encountered in medical and biomedical studies.

Methods: We conducted an extensive Monte Carlo simulation study to compare the four estimation methods: MM, MLE, MCE-EQ, and MCE-RP. For each method, we generated independent and identically distributed samples from a standard Laplace distribution (μ=0, b=1) with sample sizes n = 50, 100, 200, and 400. Each scenario was replicated 1,000 times. The performance of each estimator was assessed using the root mean squared error (RMSE) for both μ and b. The MCE-RP method utilized pre-computed Representative Points for the standard Laplace distribution, which were transformed according to preliminary MLE estimates to form an optimal cell structure for chi-square minimization. All nonlinear optimizations required for MCE-EQ and MCE-RP were implemented programmatically.

Results: The simulation results indicate that MLE performs best for estimating the location parameter μ across all sample sizes. However, for the scale parameter b, the MCE-RP method consistently yields lower RMSE values compared to MLE, MM, and MCE-EQ. In many cases, particularly for smaller samples, the RMSE of MCE-RP is approximately half that of MLE for b. The advantage of MCE-RP is evident across varying numbers of Representative Points (m = 5, 10, 15, 20), with optimal performance often observed at m = 10 or 15. These findings confirm that MCE-RP provides a more precise and reliable estimator for the scale parameter, making it particularly advantageous in small-sample settings.

Contribution: This paper contributes to the statistical methodology for the Laplace distribution by introducing and validating the use of Representative Points within a minimum chi-square estimation framework. The key contributions are: (1) demonstrating that MCE-RP significantly outperforms established methods for estimating the scale parameter; (2) providing empirical evidence that RP-based discretization enhances estimation efficiency, especially in finite samples; (3) offering practical guidance for applied researchers in fields such as medical statistics, where accurate scale estimation is crucial for variability assessment, power analysis, and reliable inference; and (4) laying a methodological foundation for extending the RP approach to other location-scale distributions.

Keywords: Laplace distribution, Minimum chi-square estimation, Representative points.

INTRODUCTION

The Laplace (or double exponential) distribution, with its characteristic sharp peak and heavier tails compared to the Gaussian, has long transcended its origins in Laplace's work on error theory to become a vital tool across scientific disciplines. In economics and finance, it effectively models asset returns and extreme market movements. In engineering, it describes certain types of noise and signals. In the life sciences, particularly medical and biological research, it emerges in contexts where data exhibit robust central tendencies alongside pronounced outliers or asymmetric variations—common in measurements like biomarker concentrations, growth rates, or physiological response times [1]. Its mathematical elegance, defined by a location parameter (μ, the median and mode) and a scale parameter ($b > 0$), facilitates analytical tractability while providing a more

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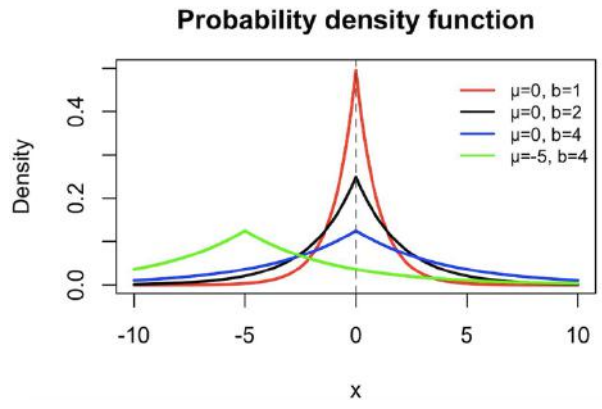
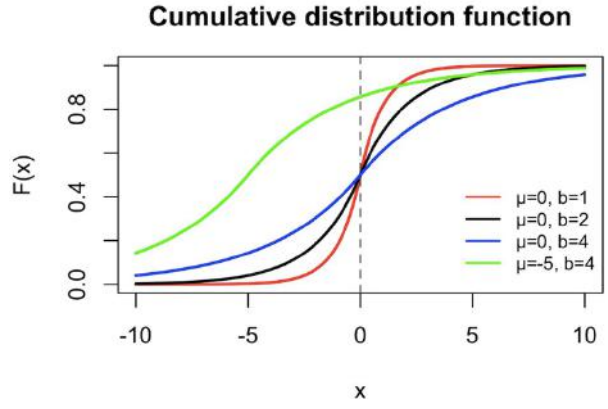
Parameters	μ, location (real) $b > 0$, scale (real)
Support	$\mathbb{R}$=$(-\infty, +\infty)$
PDF	$f(x; \mu, b) = \frac{1}{2\mathrm{b}} \exp\left(-\frac{\lvert \mathrm{x} - \mu \rvert}{b}\right)$
CDF (cumulative distribution function)	$\begin{cases} \frac{1}{2}\exp\left(\frac{\mathrm{x} - \mu}{b}\right), & if\ x \le \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{\mathrm{x} - \mu}{b}\right), & if\ x \ge \mu \end{cases}$
Mean	μ
Median	μ
Mode	μ
Variance	$2b^2$
	

realistic error model than the normal distribution for many real-world processes characterized by "spikiness" and "burstiness." Here is a simple summary on the basic properties of the Laplace distribution.

The utility of the Laplace model hinges on two foundational statistical tasks: accurate parameter estimation and rigorous assessment of model adequacy. For estimation, practitioners require methods that are not only theoretically sound but also efficient and robust across varying sample sizes. While Maximum Likelihood Estimation (MLE) is the gold standard for the Laplace distribution, offering consistency and asymptotic efficiency, its performance—particularly in terms of finite-sample bias and variance—can be suboptimal when dealing with the modest sample sizes frequently encountered in specialized medical studies or pilot trials. Alternative methods, such as method of moments (MM) or various chi-square techniques, offer different trade-offs between robustness, simplicity, and efficiency, yet a systematic, empirical comparison of these contenders, especially under small-sample conditions, remains a pertinent research gap.

A sophisticated approach to bridging parametric estimation and nonparametric testing lies in the concept of Representative Points (RPs) or Principal Points [2, 3, 4]. RPs are a finite set of points optimally selected to represent a continuous probability distribution, minimizing a quantizer error such as the mean squared error (MSE) [5]. This concept, rooted in information theory and numerical analysis, has found applications in numerical integration and stochastic optimization [6]. Statistically, a set of RPs and their associated probabilities can be used to construct a discrete approximation of a continuous distribution. This discretization provides a natural and statistically efficient framework for designing chi-square distance: the RPs can serve as the midpoints or "representative" values for constructing cells, and their associated probabilities define the expected cell frequencies.

This RP-based discretization is not arbitrary; it is optimal in capturing the distribution's shape. Consequently, the chi-square distance built on RP cells are hypothesized to be more sensitive to discrepancies between the empirical data and the theoretical model, as the cells are aligned with the distribution's inherent structure rather than a simple probability-equalizing rule. Furthermore, the objective function used to select RPs—often a form of distance minimization—can be repurposed to define a minimum chi-square estimation criterion, potentially leading to parameter estimators with favorable small-sample properties.

This paper makes a dual contribution to the statistical methodology for the Laplace distribution, unified by the innovative application of Representative Points in parametric estimation. We conduct an

extensive Monte Carlo simulation study to compare the finite-sample performance of four estimation methods for the parameters of the Laplace distribution. The study evaluates root mean squared error (RMSE) across a range of sample sizes and the number of RPs.

1. Method of Moments (MM);
2. Maximum Likelihood Estimation (MLE);
3. Minimum Chi-Square Estimation using traditional equiprobable cells (MCE-EQ);
4. Minimum Chi-Square Estimation using cells defined by Representative Points (MCE-RP).

The central thesis of this work is that the optimal discretization afforded by Representative Points provides a superior framework for both inferential tasks. We hypothesize that the MCE-RP estimator will be highly competitive with, and potentially superior to, MLE in small samples. Our findings aim to provide medical and applied researchers with more reliable tools for parameter estimation when using the Laplace distribution, thereby strengthening the statistical rigor of their analyses. Finally, we posit that the success of the RP methodology here signals its potential for fruitful application to the broader family of location-scale distributions. This paper is organized as follows. Section 2 presents the empirical comparisons among the four estimation methods as mentioned. Some concluding remarks are given in the last section.

2. AN EMPIRICAL COMPARISON IN PARAMETRIC ESTIMATION

2.1. Method of Moment (MM)

The table in section 1 gives the Laplace population moments. The first population moment is just the expected value $E(X) = \mu$. The second population moment $E(X^2) = var(X) + (EX)^2 = 2b^2 + \mu^2$. Based on the principle of the method of moment (MM), the MM estimators for μ and b are solutions to the following equations from an i.i.d. sample $\{X_1, \dots, X_n\}$ from the Laplace population:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}, \qquad 2b^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2, \tag{1}$$

Which give the MM estimators for μ and b as

$$\hat{\mu}_{mm} = \bar{X}, \quad \hat{b}_{mm} = \frac{1}{\sqrt{2}} S_n, \; S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{2}$$

2.2. The Maximum Likelihood Estimation (MLE)

Define the log-likelihood function:

$$l(x;\, \mu, b) = -n[\log(2b)] - \frac{1}{b}\sum_{i=1}^{n} |X_i - \mu| \tag{3}$$

It is well-known that maximization of equation (3) is

$$\hat{\mu}_{mle} = median\{X_1, \dots, X_n\}, \qquad \hat{b}_{mle} = \frac{1}{n}\sum_{i=1}^{n} |X_i - \hat{\mu}_{mle}|. \tag{4}$$

These are the MLEs for μ and b, respectively.

2.3. The Minimum Chi-square Estimator under Equiprobable Cells

Write the probability density function (PDF) of the Laplace distribution as

$$f(x; \mu, b) = \frac{1}{b} f_0\left(\frac{x-\mu}{b}\right), \;\; f_0(t) = \frac{1}{2} exp(-|t|), -\infty < t < +\infty \tag{5}$$

where $f_0(t)$ is the PDF of the standard Laplace distribution. For any given positive integer *m*, let constants $c_1, \dots, c_{m-1}$ be determined by

$$\int_{c_{j-1}}^{c_j} \frac{1}{\sigma} f_0(y)dy = \frac{1}{m}, j = 1, \dots, m, c_0 = -\infty, c_m = +\infty \tag{6}$$

Define the cells:

$$J_j = (\hat{\mu}_{mle} + \hat{b}_{mle} c_j \quad , \quad \hat{\mu}_{mle} + \hat{b}_{mle} c_{j+1}), j = 1, \dots, m, \tag{7}$$

and the cell probabilities:

$$p_i(\mu, b) = \int_{J_i} \frac{1}{b} f_0\left(\frac{x-\mu}{b}\right) dx, \;\; i = 1, \dots, m \tag{8}$$

Define the intervals $\{I_i = (a_{i-1}, a_i): i = 1, \dots, m, a_0 = -\infty, a_m = +\infty\}$ with a_i given by

$$\int_{a_{i-1}}^{a_i} f_0(t)dt = \frac{1}{m}, \;\; i = 1, \dots, m \tag{9}$$

Let $n_i \; (i = 1, \dots, m)$ be the observed frequency that the sample points $\{X_1, \dots, X_n\}$ fall inside the interval $I_i = (a_{i-1}, a_i)$, $\sum_{i=1}^{m} n_i = n$. The Pearson chi-square distance between the observed frequency n_i and the expected frequency $e_i = np_i(\mu, b)$ is defined by

$$X_n^2(\mu, b) = \sum_{i=1}^{m} \frac{(n_i - np_i(\mu,b))^2}{np_i(\mu,b)}, \tag{10}$$

where $p_i(\mu, b)$ is given by (8). The equiprobable minimum chi-square estimator (MCE-EQ) for *μ* and *b* are the point (μ, b) that minimizes (10). That is, the MCE-EQ for *μ* and *b* are the solutions to the following equation

$$\frac{\partial}{\partial \theta_j} X_n^2(\boldsymbol{\theta}) = -2\sum_{i=1}^{m}\left\{\frac{n_i - np_i(\boldsymbol{\theta})}{p_i(\boldsymbol{\theta})} + \frac{(n_i - np_i(\boldsymbol{\theta}))^2}{2np_i(\boldsymbol{\theta})}\right\}\frac{\partial}{\partial \theta_j} p_i(\boldsymbol{\theta}) = 0, j = 1, 2 \tag{11}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2) = (\mu, b)$. Fisher (1924) [7] commented that for a relatively large sample size *n*,

$$\sum_{i=1}^{m} \frac{(n_i - np_i(\boldsymbol{\theta}))^2}{2np_i(\boldsymbol{\theta})} \to 0, \text{ in probability.}$$

As a result, equation (11) is approximately equivalent to

$$\frac{\partial}{\partial\theta_j}X_n^2(\theta)=0\ approximately\ \Rightarrow$$
$$\sum_{i=1}^{m}\frac{n_i-np_i(\boldsymbol{\theta})}{p_i(\boldsymbol{\theta})}\frac{\partial}{\partial\theta_j}p_i(\boldsymbol{\theta})=0 \quad (12)$$

The solution $(\hat{\mu}_{mce-eq},\hat{b}_{mce-eq})$ is called the MCE-EQ for μ and b. It is obvious that equation (12) is a nonlinear equation that has to be solved by computer programs like R, Python, or MATLAB.

2.4. The Minimum Chi-square Estimator under RP Cells

A set of RPs $\{R_i: i=1,\dots,m\}$ for a continuous probability distribution with PDF $f(x)$ is a set of points that minimize the mean-squared-error (MSE) defined by the function [2]:

$$L(x_1,\dots,x_m)=\frac{1}{\mathrm{Var}(X)}\int_{-\infty}^{\infty}\min_{1\le i\le m}(x-x_i)^2f(x)\,dx \quad (13)$$

The corresponding weight p_i for each RP R_i is the probability given by

$$p_i=\int_{(R_{i-1}+R_i)/2}^{(R_i+R_{i+1})/2}f(x)\,dx\,,i=1,\dots,m,R_0=-\infty,R_{m+1}=+\infty. \quad (14)$$

The following website provides the RPs $\{R_1,\dots,R_m\}$ for the standard Laplace distribution $(\mu=0,b=1)$ and their associated probabilities $\{p_1,\dots,p_m\}$ as defined by (14) for a number of standard location-scale probability distributions:

https://fst.uic.edu.cn/isci_en/Representative_Points/MSE_Representative_Points_for_Different_Statistica.htm.

For the location-scale Laplace distribution in the table in Section 1, the estimated RPs are obtained by

$$\hat{R}_i=\hat{\mu}_{mle}+\hat{b}_{mle}R_i,\ \ i=1,\dots,m \quad (15)$$

The weight $\hat{p}_i$ for $\hat{R}_i$ is equal to the weight p_i for R_i because

$$\hat{p}_i=\int_{\frac{\hat{R}_{i-1}+\hat{R}_i}{2}}^{\frac{\hat{R}_i+\hat{R}_{i+1}}{2}}\frac{1}{\hat{b}_{mle}}f_0\left(\frac{x-\hat{\mu}_{mle}}{\hat{b}_{mle}}\right)dx=$$
$$\int_{\hat{\mu}_{mle}+\hat{b}_{mle}\frac{R_{i-1}+R_i}{2}}^{\hat{\mu}_{mle}+\hat{b}_{mle}\frac{R_i+R_{i+1}}{2}}\frac{1}{\hat{b}_{mle}}f_0\left(\frac{x-\hat{\mu}_{mle}}{\hat{b}_{mle}}\right)dx=\int_{\frac{R_{i-1}+R_i}{2}}^{\frac{R_i+R_{i+1}}{2}}f_0(y)dy=p_i \quad (16)$$

for $i=1,\dots,m,R_0=-\infty,R_{m+1}=+\infty$. Similar to equations (11)-(12), the RP minimum chi-square estimator (MCE-RP) for μ and b are the solutions to equation (12) with

$$p_i(\mu,b)=\int_{K_i}\frac{1}{b}f_0\left(\frac{x-\mu}{b}\right)dx,\ K_i=\left(\frac{R_{i-1}+R_i}{2},\frac{R_i+R_{i+1}}{2}\right),i=1,\dots,m\,,R_0=-\infty,R_{m+1}=+\infty \quad (17)$$

The solution $(\hat{\mu}_{mce-rp},\hat{b}_{mce-rp})$ is called the MCE-RP for μ and b. It is true that equation (12) with $p_i(\mu,b)$ given by (17) is also a nonlinear equation that has to be solved by computer programs like R, Python, or MATLAB.

2.5. Monte Carlo Comparison

This subsection presents the above four estimation methods in subsections 2.1-2.4. We use the MATLAB code to generate the i.i.d. samples for a given set of (μ,b) and then implement the methods in subsections 2.1-2.4 to obtain the four types of estimators for (μ,b). The Monte Carlo experiments were carried out for 1,000 replications because of the time to solve the nonlinear equations for the MCE-EQ and MCE-RP estimators. The Root Mean Square Error (RMSE) is employed to evaluate the accuracy between each type of estimator and its true value. The RMSE is defined by

$$RMSE(\hat{\theta}_j)=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_j-\theta_0)^2} \quad (18)$$

where $\hat{\theta}_j$ denotes the estimator for μ or b, θ_0 denotes the true value of μ or b, correspondingly, N stands for the number of Monte Carlo replications. The Monte Carlo outcomes under 1,000 replications are given in Tables **1-4** under sample sizes ranging from n=50 to n=400, where the estimates for the location and scale parameters μ and b are the mean values from the 1,000 replications, the columns Vs. MLE1= $\mathrm{RMSE}(\hat{\mu})/\mathrm{RMSE}(\hat{\mu}_{mle})$, and Vs. MLE2= $\mathrm{RMSE}(\hat{b})/\mathrm{RMSE}(\hat{b}_{mle})$. This means that each estimation method is compared with the MLE. At the two columns Vs. MLE1 and Vs. MLE2, the smaller the number at these two columns, the better the average performance of the estimation method. The following empirical conclusions can be summarized:

1) For the location parameter, the MLE has the best average performance in the sense of the RMSE for the sample sizes ranging from n=50 to n=400. The MM estimate and the MCE-EQ estimates are generally worse than the MLE. but the MCE-RP is comparable with the MLE with Vs. MLE1 numbers close to 1. MCE-RP could improve the estimation for the location parameter if the number of RPs is appropriately chosen, for example, for n=50 in Table **1** and n=100 in Table **2**, the numbers (.6579 and .9175, respectively) at the column Vs. MLE1 are less than 1 for the number of RPs m=10, indicating that the RP minimum chi-square estimate MCE-RP2 could improve the MLE significantly;

Table 1: Monte Carlo Comparison Among Four Estimation Methods

(Sample Size *n*=50, True Values $(\mu, b) = (0,1)$)

Method	$\hat{\mu}$	$\hat{b}$	RMSE($\hat{\mu}$)	Vs. MLE1	RMSE($\hat{b}$)	Vs. MLE2
MM	.0021	.9852	.2020	1.4018	.1585	.9950
MLE	.0068	.9928	.1441	1	.1593	1
MCE-EQ1	.0361	.9913	.2250	1.5614	.3215	2.0182
MCE-EQ2	.0897	.7955	.1827	1.2679	.3034	1.9046
MCE-RP1	.0335	.9611	.1951	1.3539	.1316	.8261
MCE-RP2	.0014	1.0104	.0948	.6579	.1195	.7502
MCE-RP3	.0325	.9632	.1487	1.0319	.1459	.9159
MCE-RP4	.0054	1.0040	.1901	1.3192	.0853	.5355

Note:
1. MCE-EQ1 corresponds to the number of cells $k = 4$, MCE-EQ2 ($k = 9$);
2. MCE-RP1 corresponds to the number of RPs $m = 5$, MCE-RP2 ($m = 10$), MCE-RP3 ($m = 15$), MCE-RP4 ($m = 20$).

2) For the scale parameter, the RP minimum chi-square estimate MCE-RP has the best average performance in the sense of the RMSE for the sample sizes ranging from *n*=50 to *n*=400. The numbers at the column Vs. MLE2 for MCE-RP are all smaller than 1, indicating that the RP minimum chi-square estimate MCE-RP always improves the MLE. The RMSE for MCE-RP could be around half of the RMSE for MLE, implying MCE-RP estimate for the scale parameter could reduce the RMSE by around 50% compared to MLE.

In small samples (*n*=50), MCE-RP with *m*=10 reduces the RMSE for the scale parameter by approximately 25% compared to MLE, demonstrating its practical advantage in pilot studies or rare disease research where sample sizes are limited.

Table 2: Monte Carlo Comparison Among Four Estimation Methods

(Sample Size *n*=100, True Values $(\mu, b) = (0, 1)$)

Method	$\hat{\mu}$	$\hat{b}$	RMSE($\hat{\mu}$)	Vs. MLE1	RMSE($\hat{b}$)	Vs. MLE2
MM	.0008	.9946	.1312	1.3360	.1103	1.0204
MLE	.0079	.9910	.0982	1	.1081	1
MCE-EQ1	.00457	.8938	.1942	1.9776	.2343	2.1674
MCE-EQ2	.0170	1.0616	.1101	1.1212	.2258	2.0888
MCE-RP1	.0515	.9785	.1469	1.4959	.0724	.6698
MCE-RP2	.0089	.9854	.0901	.9175	.0985	.9112
MCE-RP3	.0232	1.0029	.1005	1.0234	.1031	.9537
MCE-RP4	.0165	.9864	.1104	1.1242	.1045	.9667

Table 3: Monte Carlo Comparison Among Four Estimation Methods

(Sample Size *n*=200, True Values $(\mu, b) = (0, 1)$)

Method	$\hat{\mu}$	$\hat{b}$	RMSE($\hat{\mu}$)	Vs. MLE1	RMSE($\hat{b}$)	Vs. MLE2
MM	.0006	.9951	.0993	1.3678	.0809	1.0715
MLE	.0011	.9940	.0726	1	.0755	1
MCE-EQ1	.0391	.9734	.1372	1.8898	.1217	1.6119
MCE-EQ2	.0144	.9578	.1648	2.2700	.0969	1.2834
MCE-RP1	.0362	.9903	.0803	1.1061	.0685	.9073
MCE-RP2	.0064	1.0011	.0864	1.1901	.0459	.6079
MCE-RP3	.0301	.9800	.0754	1.0386	.0391	.5179
MCE-RP4	.0080	1.0341	.0634	.8733	.0664	.8795

Table 4: Monte Carlo Comparison Among Four Estimation Methods

(Sample Size *n*=400, True Values $(\mu, b) = (0, 1)$)

Method	$\hat{\mu}$	$\hat{b}$	RMSE($\hat{\mu}$)	Vs. MLE1	RMSE($\hat{b}$)	Vs. MLE2
MM	.0007	.9972	.0702	1.4182	.0539	1.0228
MLE	.0022	.9981	.0495	1	.0527	1
MCE-EQ1	.0483	.9793	.0620	1.2525	.1112	2.1100
MCE-EQ2	.0019	1.0174	.1156	2.3354	.0413	.7837
MCE-RP1	.0173	1.0057	.0446	0.9010	.0501	.9507
MCE-RP2	.0137	.9822	.0563	1.1374	.0517	.9810
MCE-RP3	.0426	1.0181	.0510	1.0303	.0416	.7894
MCE-RP4	.0484	1.0048	.0633	1.2788	.0517	.9810

For moderate samples, MCE-RP continues to outperform MLE in scale estimation, with RMSE reductions up to 33% (MCE-RP1, *m*=5 RPs), supporting its use in clinical trials where precise variability estimation is essential for power calculations.

As sample size increases, MCE-RP maintains superior scale estimation efficiency, with RMSE values nearly half (MCE-RP3, m=15 RPs) of those from MLE in some configurations, reinforcing its robustness in larger biomedical datasets.

Even in larger samples, MCE-RP remains competitive, particularly for scale estimation, suggesting its utility in meta-analyses or population-level medical studies where accurate dispersion estimation is key to pooled effect size interpretation.

3. CONCLUDING REMARKS

This study has introduced and empirically validated the use of Representative Points (RPs) within a minimum chi-square estimation framework for the Laplace distribution. Our Monte Carlo simulations demonstrate that the RP-based minimum chi-square estimator (MCE-RP) significantly outperforms traditional methods—including Maximum Likelihood Estimation (MLE)—for estimating the scale parameter, particularly in small to moderate samples. While MLE remains robust for location parameter estimation, MCE-RP provides a more precise and reliable estimator for the scale parameter, often reducing RMSE by up to 50% compared to MLE. This improvement stems from the optimal discretization of the distribution afforded by Representative Points, which aligns the chi-square distance metric with the inherent structure of the underlying probability model.

The primary methodological contribution of this work is the demonstration that an information-theoretically optimal discretization—via Representative Points—can substantially enhance parameter estimation within a minimum distance framework. By bridging numerical approximation theory and statistical inference, MCE-RP offers a principled alternative to likelihood-based methods, especially valuable in finite-sample settings. The approach is generalizable and lays a foundation for extending RP-based estimation to other location-scale distributions, such as the logistic, Student's t, or generalized extreme value families.

Medical research implications can be summarized as follows. Accurate estimation of variability is critical in medical and biomedical research, where the scale parameter governs dispersion in key measures such as biomarker concentrations, physiological response times, or treatment effect heterogeneity. The superior performance of MCE-RP for scale estimation directly supports more reliable sample size calculations, improved risk assessment, robust diagnostic threshold determination, and enhanced evaluation of therapeutic efficacy. In practice, this can contribute to better-powered studies, more reproducible findings, and clinically actionable inference—particularly in pilot studies, rare disease research, and clinical trials where sample sizes are often limited.

Several limitations should be acknowledged. First, the performance of MCE-RP depends on the appropriate selection of the number of Representative Points (*m*); our simulations suggest *m* = 10 or 15 as generally effective, but context-specific tuning may be warranted. Second, the method involves nonlinear optimization and is computationally more intensive than MLE or MM, though remains feasible with modern computational tools. Third, the approach assumes correct specification of the Laplace model; performance under model misspecification or in the presence of outliers warrants further investigation. Finally, while this study focuses on the standard Laplace distribution, future work should explore

extensions to contaminated data, weighted distributions, or heteroscedastic settings commonly encountered in medical applications.

Based on our findings, we strongly recommend the use of MCE-RP for estimating the scale parameter of the Laplace distribution, especially in small to moderate samples. For the location parameter, MLE remains a reliable choice, though MCE-RP can offer comparable performance with appropriate RP selection. Researchers in medical statistics and applied fields are encouraged to adopt RP-based estimation when modeling data with Laplace characteristics, as it provides a more accurate and efficient tool for variability assessment—a cornerstone of rigorous scientific inference.

In summary, this research illustrates that Representative Points are more than a numerical convenience; they are a potent methodological asset that enhances statistical estimation and inference. By integrating optimal discretization into estimation procedures, MCE-RP offers a robust, efficient alternative to conventional methods, with meaningful implications for the precision and reliability of statistical analysis in medical research and beyond.

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