A Weak form of Hadwiger's Conjecture

Dominic van der Zypen*

M&S Software Engineering, Morgenstrasse 129, CH-3018 Bern, Switzerland

Abstract: We introduce the following weak version of Hadwiger's conjecture: If \( G \) is a graph and \( \kappa \) is a cardinal such that there is no coloring map \( c : G \to \kappa \), then \( K_\kappa \) is a minor of \( G \). We prove that this statement is true for graphs with infinite chromatic number.

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1. DEFINITIONS

In this note we are only concerned with simple undirected graphs \( G = (V,E) \) where \( V \) is a set and \( E \subseteq \mathcal{P}_2(V) \) where

\[
\mathcal{P}_2(V) = \{ \{x,y\} : x,y \in V \text{ and } x \neq y \}.
\]

We denote the vertex set of a graph \( G \) by \( V(G) \) and the edge set by \( E(G) \). Moreover, for any cardinal \( \alpha \) we denote the complete graph on \( \alpha \) points by \( K_\alpha \).

For any graph \( G \), disjoint subsets \( S,T \subseteq V(G) \) are said to be connected to each other if there are \( s \in S, t \in T \) with \( \{s,t\} \in E(G) \). Note that \( K_\alpha \) is a minor of a graph \( G \) if and only if there is a collection \( \{S_\beta : \beta \in \alpha \} \) of nonempty, connected, and pairwise disjoint subsets of \( V(G) \) such that for all \( \beta, \gamma \in \alpha \) with \( \beta \neq \gamma \) the sets \( S_\beta \) and \( S_\gamma \) are connected to each other.

Well-founded trees and well-founded tree decompositions as defined in [1] will be central later on:

1.1. Definition

A well-founded tree is a non-empty partially ordered set \( T = (V,\leq) \) such that for every two elements \( t_1,t_2 \) their infimum exists and the set \( \{t' \in V : t' < t \} \) is a well-ordered chain for every \( t \in V \). For \( t_1,t_2 \in V = V(T) \) we set \( T[t_1,t_2] = \{ t \in V(T) : t \geq \inf \{t_1,t_2\} \text{ and } (t \leq t_1 \text{ or } t \leq t_2) \} \).

1.2. Definition

A well-founded tree-decomposition of a graph \( G \) is a pair \((T,W)\) where \( T \) is a well-founded tree and \( W : V(T) \to \mathcal{P}(V(G)) \) is a map such that

- \( V(G) = \bigcup \text{im}(W) \), and
- \( E(G) \subseteq \bigcup \{ \{W(t) : t \in V(T) \} \}
- \( \text{if } t' \in T[t_1,t_2] \text{ then } W(t_1) \cap W(t_2) \subseteq W(t') \)
- \( \text{if } C \subseteq V(T) \text{ is a chain with } c = \sup C \in V(T) \text{, then } C \cap \{W(t) : t \in C\} \subseteq W(c) \).

Note that (W1) says that every vertex of \( G \) is contained in some \( W(t) \), and every edge has both its endpoints in some \( W(t) \).

1.3. Definition

We say that a well-founded tree-decomposition has width \( < \kappa \) if for every chain \( C \subseteq V(T) \) we have

\[
\text{card}\left( \bigcup_{t \in C} W(t') : t' \in C, t' \geq t \right) < \kappa.
\]

For the singleton chain \( C = \{t\} \) this implies

\[
\text{card}(W(t)) < \kappa \text{ for every } t \in V(T).
\]

2. THE WEAK HADWIGER CONJECTURE

In [2], Hadwiger formulated his well-known and deep conjecture, linking the chromatic number \( \chi(G) \) of a graph \( G \) with clique minors. He conjectured that if \( \chi(G) = n \) then \( K_n \) is a minor of \( G \). However, for graphs with infinite chromatic number, the conjecture does not hold: in [3] a graph \( G \) is given such that \( \chi(G) = \omega \), but \( K_\omega \) is not a minor of \( G \).

We consider the following weaker form of Hadwiger's conjecture:

**Weak Hadwiger Conjecture**

Let \( G \) be a graph and \( \kappa \) be a cardinal such that there is no coloring map \( c : G \to \kappa \). Then \( K_\kappa \) is a minor of \( G \).
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Note that in the finite case, this statement translates to: if \( \chi(G) = n \) then \( K_{n-1} \) is a minor of \( G \). As of now, it seems to be an open problem whether the weak Hadwiger conjecture is true in the finite case.

However in the infinite case, we can use the following structure theorem by Robertson, Seymour, and Thomas:

2.1. Theorem

Let \( \kappa \) be an infinite cardinal and let \( G \) be a graph. Then the following two conditions are equivalent:

1. \( G \) contains no subgraph isomorphic to a subdivision of \( K_{\kappa} \);
2. \( G \) admits a well-founded tree-decomposition of width \( <\kappa \).

The strategy is the following. We fix any graph \( G \) and cardinal \( \kappa \) and assume that \( K_{\kappa} \) is not a minor of \( G \). Then we construct a \( \kappa \)-coloring of \( G \).

If \( K_{\kappa} \) is not a minor of \( G \), it is not a topological minor of \( G \), which is equivalent to condition (1) of structure. So we apply theorem structure and it remains to prove the following statement:

2.2. Proposition

Let \( G \) be a graph with a well-founded tree-decomposition of width \( <\kappa \). Then there is a coloring map \( c:G \rightarrow \kappa \).

Proof. It is sufficient to construct a mapping \( f:V(G) \rightarrow \kappa \) such that the restriction \( f|_{W(t)} \) is injective for every \( t \in T \) since every edge lies entirely in some \( W(t) \), the function \( f \) will be a coloring of \( G \).

We set \( X = V(G) \). Denote the ordering relation on \( T \) by \( \leq_{T} \). It is easy to see that \( \leq_{T} \) can be extended to a total well-ordering \( \leq_w \) on \( T \). Moreover, for \( x \in X \) we define

\[
m(x) = \min\{t \in T : x \in W(t)\},
\]

where the minimum is taken with respect to the well-ordering \( \leq_w \) on \( T \). (Note that the minimum is taken over a non-empty set since \( X = \bigcup_{t \in T} W(t) \).) For \( t \in T \) let \( q(t): W(t) \rightarrow \text{card}(W(t)) < \kappa \) be a bijection.

Endow \( X \) with a total well-ordering relation \( \leq_x \) defined by

\[
x \leq_x y \Leftrightarrow m(x) \leq m(y) \text{ or } [m(x) = m(y) \text{ and } \varphi_{m(x)}(x) \leq \varphi_{m(y)}(y)].
\]

We define \( f:X \rightarrow \kappa \) recursively by

\[
f(x) = \min(\kappa \cap \{f(z) : z \leq_x x \text{ and } z \in W(m(x))\}).
\]

Note that the minimum above exists since \( \kappa > \text{card}(W(t)) \) for all \( t \in T \).

It remains to show that for \( t_0 \in T \) and \( a \neq b \in W(t_0) \) we have \( f(a) \neq f(b) \). Take any \( a \leq_x b \in W(t_0) \). We consider the tree elements \( m(a), m(b) \in T \). If \( m(a) = m(b) \) then by the very definition of \( f \) we get \( f(a) = f(b) \) directly.

So suppose that \( m(a) \neq m(b) \). If \( m(b) \leq_T t_0 \) then consider \( i = \inf\{m(b), t_0 \} \) in the tree. Clearly \( i < m(b) \) and because of axiom (W2) we have \( b \in W(m(b)) \cap W(t_0) \subseteq W(i) \), which contradicts the minimality of \( m(b) \). Since the same argument can be made for \( m(a) \) we get

\[
m(a), m(b) \leq_T t_0.
\]

The definition of \( \leq_x \) and the fact that \( a \leq_x b \) and \( m(a) \neq m(b) \) jointly imply \( m(a) \leq_w m(b) \). Since predecessors of \( t_0 \) are linearly ordered in \( \leq_T \) we have \( m(a) \leq_T m(b) \) or \( m(b) \leq_T m(a) \). Recall that \( \leq_w \) extends \( \leq_T \), so we get \( m(a) \leq_T m(b) \). Therefore \( m(b) \in T[m(a), t_0] \) and we can apply axiom (W2) again to get

\[
a \in W(m(a)) \cap W(t_0) \subseteq W(m(b)).
\]

Again we go back to the recursive definition of \( f \) we have \( f(b) = \min(\kappa \cap \{f(z) : z \leq_x b \text{ and } z \in W(m(b))\}) \), and we get \( f(b) \neq f(a) \) from the fact that \( a \in W(m(b)) \).

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REFERENCES


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