A Removability Result for Holomorphic Functions of Several Complex Variables

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Abstract: Suppose that $\Omega$ is a domain of $\mathbb{C}^n$, $n \geq 1$, $E \subset \Omega$ closed in $\Omega$, the Hausdorff measure $\mathcal{H}^{2n-1}(E) = 0$, and $f$ is holomorphic in $\Omega \setminus E$. It is a classical result of Besicovitch that if $n = 1$ and $f$ is bounded, then $f$ has a unique holomorphic extension to $\Omega$. Using an important result of Federer, Shiffman extended Besicovitch's result to the general case of arbitrary number of several complex variables, that is, for $n \geq 1$. Now we give a related result, replacing the boundedness condition of $f$ by certain integrability conditions of $f$ and replacing $\frac{\partial^j f}{\partial x_j}$ by $\frac{\partial^j f}{\partial x_j}$.

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1. INTRODUCTION

1.1. Previous Results

The following result of Besicovitch is well-known:

**Theorem 1.** ([1], Theorem 1, p. 2) Let $D$ be a domain in $\mathbb{C}$. Let $E \subset D$ be closed in $D$ and let $\mathcal{H}^k(E) = 0$. If $f : D \setminus E \to \mathbb{C}$ is holomorphic and bounded, then $f$ has a unique holomorphic extension to $D$.

Above and below $\mathcal{H}^k$ is the $\alpha$-dimensional Hausdorff (outer) measure in $\mathbb{R}^k$, $k \geq 2$.

Much later Shiffman gave the following general result:

**Theorem 2.** ([2], Lemma 3, p. 115) Let $\Omega$ be a domain in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in $\Omega$ and let $\mathcal{H}^{2n-1}(E) = 0$. If $f : \Omega \setminus E \to \mathbb{C}$ is holomorphic and bounded, then $f$ has a unique holomorphic extension to $\Omega$.

Shiffman's proof was based on Besicovitch's result, Theorem 1 above, on coordinate rotation, on the use of Cauchy integral formula and on the following result of Federer:

**Lemma 1.** ([3], Theorem 2.10.25, p. 188, and [2], Corollary 4, Lemma 2, p. 114) Suppose that $E \subset \mathbb{R}^k$, $k \geq 2$, is such that $\mathcal{H}^{k-1}(E) = 0$. Then for all $j$, $1 \leq j \leq k$, and for $\mathcal{H}^{k-1}$-almost all $X_j \in \mathbb{R}^{k-1}$ the set $E(X_j)$ is empty.

For slightly more general versions of Shiffman's result with different proofs, see [4], Theorem 3.1, p. 49, Corollary 3.2, p. 52, and [5], Theorem 3.1, p. 333, Corollary 3.3, p. 336.

1.2. Notation

Our notation is more or less standard, see [6-8]. However and for the convenience of the reader, we recall here the following. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $j \in \mathbb{N}$, $1 \leq j \leq n$, then we write $x = (x_j, X_j)$, where $X_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$. Moreover, if $E \subset \mathbb{R}^n$, $1 \leq j \leq n$, and $x^0_j \in \mathbb{R}$, $X^0_j \in \mathbb{R}^{n-1}$, we write

$E(x^0_j) = \{X_j \in \mathbb{R}^{n-1} : x = (x^0_j, X_j) \in E\}$,

$E(X^0_j) = \{x_j \in \mathbb{R} : x = (x_j, X^0_j) \in E\}$.

If $\Omega \subset \mathbb{R}^n$ and $p > 0$, then $L^p_{\text{loc}}(\Omega)$, $p > 0$, is the space of functions $u$ in $\Omega$ for which $|u|^p$ is locally integrable on $\Omega$. We identify $\mathbb{C}^n$, $n \geq 1$, with $\mathbb{R}^{2n}$. We use the common convention $0 \cdot \pm \infty = 0$.

For the definition and properties of subharmonic functions, see e.g. [9-12], for the definition of holomorphic functions see e.g. [13-15].
2. AN EXTENSION RESULT FOR HOLOMORPHIC FUNCTIONS

2.1. Our result is related to Theorem 2 above, and reads as follows:

Theorem 3. Suppose that \( \Omega \) is a domain in \( \mathbb{C}^n \), \( n \geq 1 \). Let \( E \subset \Omega \) be closed in \( \Omega \) and let \( \mathcal{H}^{2n-1}(E) = 0 \). Let \( f : \Omega \setminus E \to \mathbb{C} \) be holomorphic and such that the following conditions are satisfied:

(i) \( f \in \mathcal{L}^n_\text{loc}(\Omega) \),

(ii) for each \( j, 1 \leq j \leq 2n \), \( \frac{\partial^2 f}{\partial x_j^2} \in \mathcal{L}^n_\text{loc}(\Omega) \).

Then \( f \) has a holomorphic extension to \( \Omega \).

2.2. The proof will be based, in addition to Federer’s cited Lemma 1 above, also on the following recent result:

Lemma 2. ([8], Theorem, p. 568) Suppose that \( \Omega \) is a domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( E \subset \Omega \) be closed in \( \Omega \) and let \( \mathcal{H}^{n-1}(E) < +\infty \). Let \( u : \Omega \to [-\infty, +\infty] \) be such that the following conditions are satisfied:

(i) \( u \in \mathcal{L}^n_\text{loc}(\Omega) \);

(ii) \( u \in \mathcal{C}^2(\Omega \setminus E) \);

(iii) for each \( j, 1 \leq j \leq n \), \( \frac{\partial^2 u}{\partial x_j^2} \in \mathcal{L}^n_\text{loc}(\Omega) \);

(iv) for each \( j, 1 \leq j \leq n \), and for \( \mathcal{H}^{n-1} \)-almost all \( X_j \in \mathbb{R}^{n-1} \) such that \( E(X_j) \) is finite, the following condition holds: for each \( x_0^j \in E(X_j) \) there exist sequences \( x_{i,j}^0, x_{j,i}^0 \in (\Omega \setminus E)(X_j), i = 1, 2, \ldots \), such that

(iv(a) \( x_{i,j}^0 \nearrow x_j^0, x_{j,i}^0 \searrow x_j^0 \), and \( \lim_{i \to \infty} (x_{i,j}^0, X_j) = \lim_{i \to \infty} (x_{j,i}^0, X_j) \in \mathbb{R} \),

(iv(b) \( \lim_{i \to \infty} \frac{\partial u}{\partial x_j}(x_{i,j}^0, X_j) \leq \lim_{i \to \infty} \frac{\partial u}{\partial x_i}(x_{j,i}^0, X_j) < +\infty \),

(v) \( u \) is subharmonic in \( \Omega \setminus E \).

Then \( u \mid (\Omega \setminus E) \) has a subharmonic extension to \( \Omega \).

Proof of Theorem 3. Write \( f = u + iv \). It is sufficient to show that \( u \) and \( v \) have subharmonic extensions to \( \Omega \). As a matter of fact, then \( f \) will be locally bounded in \( \Omega \), and thus the claim will follow from Theorem 2 or also from the already cited slightly more general results from [4, 5]. To see that \( u \) and \( v \) have indeed subharmonic extensions to \( \Omega \), we use our Lemma 2 as follows.

It is sufficient to show that the assumption (iv) of Lemma 2 is satisfied. For that purpose take \( j, 1 \leq j \leq 2n \), arbitrarily. By Federer’s result, Lemma 1 above, we know that for \( \mathcal{H}^{2n-1} \) almost all \( X_j \in \mathbb{R}^{2n-1} \) the set \( E(X_j) \) is empty. Thus for \( \mathcal{H}^{2n-1} \) almost all \( X_j \in \mathbb{R}^{2n-1} \) the functions \( u(\cdot, X_j) : \Omega(X_j) \to \mathbb{R} \) and \( v(\cdot, X_j) : \Omega(X_j) \to \mathbb{R} \) are \( C^\infty \) functions. Therefore, the assumption (iv) is satisfied both for \( u \) and for \( v \), concluding the proof.

REFERENCES


