The Gauß Sum and its Applications to Number Theory

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Abstract: The purpose of this article is to determine the monogenity of families of certain biquadratic fields \( K \) and cyclic bicubic fields \( L \) obtained by composition of the quadratic field of conductor 5 and the simplest cubic fields over the field \( Q \) of rational numbers applying cubic Gauß sums. The monogenic biquadratic fields \( K \) are constructed without using the integral bases. It is found that all the bicubic fields \( L \) over the simplest cubic fields are non-monogenic except for the conductors 7 and 9. Each of the proof is obtained by the evaluation of the partial differentials \( \zeta - \zeta' \) of the different \( \partial_{\zeta-\zeta'}(\xi) \) with \( F = K \) or \( L \) of a candidate number \( \xi \), which will or would generate a power integral basis of the fields \( F \). Here \( \rho \) denotes a suitable Galois action of the abelian extensions \( F/Q \) and \( \partial_{\zeta-\zeta'}(\xi) \) is defined by \( \prod_{\rho \in \rho_{F/Q}}(\zeta - \zeta') \), where \( G \) and \( \iota \) denote respectively the Galois group of \( F/Q \) and the identity embedding of \( F \).

Keywords: Monogenity, Biquadratic field, Simplest cubic field, Discriminant, Integral basis.

INTRODUCTION

Let \( F \) be an algebraic number field over the field \( Q \) of rational numbers with the extension degree \( n = [F:Q] \). Then the ring \( Z_F \) of integers in \( F \) has an integral basis \( \{\omega_j\}_{j=1}^{m} \) such that \( Z_F \) is the \( \mathbb{Z} \)-module \( Z \cdot \omega_1 + \cdots + Z \cdot \omega_m \) of rank \( n \). If there exists a suitable number \( \xi \in F \) such that \( Z_F = Z \cdot 1 + \cdots + Z \cdot \xi^{n-1} \), then it is said that \( Z_F \) has a power integral basis or \( F \) is monogenic. It is known as Dedekind-Hasse’s problem to determine whether an algebraic number field is monogenic or not [7, 5]. Let \( \text{Ind}_F(\xi) \) denote the index \( \left\lfloor \frac{d_F(\xi)}{d_F} \right\rfloor \) of an integer \( \xi \) in \( F \) with the discriminant \( d_F(\xi) \) of a number \( \xi \) and the field discriminant \( d_F \) of the field \( F \). This value coincides with \( \sqrt{\frac{\text{The volume of the parallelootope spanned by } (\xi^{n-1})_{1 \leq j \leq n} \times 4 \text{ (quadrants)}}{\text{The volume of the parallelootope spanned by } (\omega_i)_{1 \leq i \leq m} \times 4 \text{ (quadrants)}}} \) for \( n = 2 \). Then it is enough for the monogenity of \( F \) to find a number \( \xi \) in \( F \) such that the value \( \text{Ind}_F(\xi) \) is equal to 1. On the other hand, to show the non-monogenity we must prove that \( \text{Ind}_F(\xi) > 1 \) for every number \( \xi \) in \( F \).

Let \( \zeta_n \) be an \( n \) th root of unity and \( k_n \) be the \( n \) th cyclotomic field \( Q(\zeta_n) \) with the extension degree \( \phi(n) \), where \( \phi \) is the Euler totient function. Let \( G \) be the Galois group of \( k_n/Q \) and \( \hat{G} \) the character group of \( G \). For a character \( \chi \in \hat{G} \), the Gauß sum \( \tau_\chi \) attached to \( \chi \) is defined by the sum

\[ \sum_{x \in \mathbb{Z}} \chi(x) \zeta_n^x \]

Then \( \tau_\chi \) belongs to the field \( k_n \cdot k_n \) with the degree \( m \mid \phi(n) \) of \( \chi \). We find two phenomena.

Theorem 1.1. Let \( \lambda_\iota \) be a biquadratic character of conductor \( \iota \). Let \( K \) be a biquadratic field \( Q(\tau_{\lambda_n}, \tau_{\lambda_m}) \), where \( \tau_{\lambda_n} \) is the quadratic Gauß sum attached to \( \lambda_n \). Then

1. \( K \) is non-monogenic, if \( m = n = 1 (\text{mod } 4) \) and \( (m,n) = 1 \).
2. There exist infinitely many monogenic biquadratic fields \( K \), if \( m = 0 (\text{mod } 4) \), \( n = 1 (\text{mod } 4) \) and \( (m,n) = 1 \) or \( m = n = 0 (\text{mod } 4) \) and \( (m,n) = 4 \) or \( 8 \).

The proof is obtained without using any integral basis of a field \( Q(\tau_{\lambda_n}, \tau_{\lambda_m}) \). This result is a
Theorem 1.2. There does not exist any monogenic sextic bicubic fields $Q(\tau_{j,5})$ with the quadratic Gauss sum $\tau_{j,5}$ and the cubic Gauss period $\eta_{9\nu}$ attached to the quadratic character $\lambda_5$ and the cubic one $\nu$, with the coprime conductors 5 and $n$, respectively, where the Gauss period $\eta_{9\nu}$ is determined by $((-1)^r + \tau_{j,5} + \tau_{j,5}^3)/3$ with the cubic Gauss sum $\tau_{9\nu}$ and the number $r$ of distinct prime factors of $n$, when $n$ is square free and the fields $Q(\eta_{9\nu})$ range over the simplest cubic fields of conductor $n = a^2 + 3a + 9$ except for $n = 7$ of $a = -1$ and 9 of $a = 0$.

In the case of the prime conductor $p$ of quadratic character $\lambda_p^{\ast}$ with prime discriminant $p^{\ast} = (-1)^{p-1}/2$, $p$ and cubic one $\nu$, the monogenity of the sextic field $Q(\tau_{j,5}^{\ast}, \eta_{9\nu})$ has been determined by the first and the third authors such that there does not exist any cyclic sextic fields $Q(\tau_{j,5}^{\ast}, \eta_{9\nu})$ except for the prime power conductors 7, 3 and 13 [12].

There are related works on the abelian; pure sextic and octic extensions $F / Q$ [11, 23, 17, 15, 16, 6, 14, 3]; [4, 9, 2, 1, 8].

Proof of Theorem 1.1.

The next lemma is fundamental to simplify the proof.

Lemma 2.1. Assume that $Z_k = Z[\xi]$ for a number $\xi = \alpha + \beta \omega$ with $\alpha, \beta \in Q(\tau_{j,5})$, $\omega \in Q(\tau_{j,5})$ and field discriminants $m$ and $n$. Then

1. $\beta$ is a unit in $Q(\tau_{j,5})$.
2. $\beta$ and $\omega$ are units in $Q(\tau_{j,5})$ and $Q(\tau_{j,5})$, respectively, if $\alpha = 0$.

Proof of Lemma 2.1. Since $K = Q(\tau_{j,5}) \cdot Q(\tau_{j,5})$, there exist $\alpha, \beta \in Q(\tau_{j,5})$ and $\omega \in Q(\tau_{j,5})$ such that $\xi = \alpha + \beta \omega$. By $Ind_k(\xi) = 1$, it holds that $d_k = d_{Q(\tau_{j,5})} \cdot d_{Q(\tau_{j,5})} = d_k(\xi) = \pm N_k(\delta_k(\xi))$ with $\ell = \text{lcm}(m, n)$, where the different $\delta_k(\xi)$ of a number $\xi$ with respect to $K / Q$ is defined by $(\xi - \xi^{-\omega})(\xi^{-\omega})(\xi - \xi^{-\omega}) \cdot \text{[25]})$. The Galois group $G(K / Q)$ coincides with $(\sigma, \tau)$ with $G(Q(\tau_{j,5}) / Q) = \langle \sigma \rangle$ and $G(Q(\tau_{j,5}) / Q) = \langle \tau \rangle$, where $<\sigma, \ldots, \sigma>$ with $\sigma_j$ in $G$ means the subgroup generated by $\{\sigma_j\}_{j \in \mathbb{Z}}$ of a group $G$. Then it holds that $\sigma : \sqrt{m} \mapsto -\sqrt{m}$, $\sqrt{n} \mapsto \sqrt{n}$ and $\tau : \sqrt{n} \mapsto \sqrt{n}$.

1. Thus we have that $\xi - \xi^{-\omega} = \beta(\omega - \omega^\ast)$ and $\xi - \xi^{-\omega} = \beta(\omega - \omega^\ast)$ = $\delta_{Q(\tau_{j,5})}$, $\omega$ and $\beta$ are units in $K$.

Proof of Theorem 1.1. (1) Suppose that $Z_k = Z[\xi]$ with $\xi = \alpha + \beta \omega$, $\alpha, \beta \in Q(\tau_{j,5})$ and $\omega \in Q(\tau_{j,5})$. (i) Assume that $\alpha = 0$. Put $\beta = s + t \sqrt{m}$ and $\omega = u + \sqrt{n}$. Then by $\xi - \xi^{-\omega} = t \sqrt{m} \omega = \sqrt{m}$, $t = \pm 1$ holds. By $\xi - \xi^{-\omega} = \beta v \sqrt{m} \omega = \sqrt{n}$, $v = \pm 1$ holds. Thus it is deduced that $N_{K / Q(\tau_{j,5})}(\xi - \xi^{-\omega}) = N_{K / Q(\tau_{j,5})}(s + t \sqrt{m} \omega + \sqrt{n})$.

(2) Let $m_n$ denote the field different of an algebraic number field $M$. Since it is deduced that $\xi - \xi^{-\omega} = \beta(\omega - \omega^\ast)$ and $\xi - \xi^{-\omega} = \beta(\omega - \omega^\ast)$ = $\delta_{Q(\tau_{j,5})}$, $\omega$ and $\beta$ are units in $K$.

Therefore $K$ is non monogenic.

(2) Let $m = 4(4t - 1)$ and $n = 4(4t + 3)$ with a square free number $(4t - 1)(4t + 1)$. Then the biquadratic fields $K = Q(\tau_{j,5}, \tau_{j,5})$ coincides with $Q(\alpha, \beta)$ with $\alpha = \sqrt{m}$ and $\beta = \sqrt{n}$. Thus by the Hasse's Conductor-Discriminant Theorem, the field discriminant $d_k$ is equal to $m \cdot n \cdot m / 4 = 2^4 \cdot (4t - 1)(4t + 3)$ [25]. Choose a number $\sqrt{4t - 1 + \sqrt{4t + 3}} = \alpha + \beta$ as $\xi$. By
\( T_{K/Q}(\tau_{L_3}) (\xi) = \beta/2 \) and \( N_{K/Q}(\tau_{L_3}) (\xi) = (-\alpha^2 + \beta^2)/4 = 1 \), \( \xi \) belongs to the ring \( Z_K \) because of 
\( K \cap Z_{Q(z_{L_3})} = Z_K \), where \( Z_F \) means the integral closure of the ring \( Z_F \) of algebraic integers in a field \( F \), and for a relative field extension \( M/F \) of finite degree of algebraic number fields \( M \) and \( F \), \( T_M/F(\xi) \) and \( N_M/F(\xi) \) of a number \( \xi \) in \( M \) denote the relative norm and the relative trace, respectively. By the definition, it follows that \( d_{K/Q}(\xi) = (-1)^{4k-1} N_K(\partial_K(\xi)) \) \( = N_{K/Q}(\alpha/2; \beta/2; (\alpha+\beta)/4) = d_K \). Thus we obtain 
\( Z_K = Z[1, \xi, \xi^2, \xi^3] \).

On the cardinality of the monogenic fields \( K \) the following lemma is available.

**Lemma 2.2.** There exist infinitely many square-free numbers \( 16t^2 - 8t - 3 \) for \( t \in \mathbb{Z} \).

**Proof of Lemma 2.2** See [18, 21] or use the slightly modified Lemma 8.5 in 1st ed. of [20] with the value of \( \zeta(2) \) and prime number theorem [19]. Moreover on the density of 
\[ \#\{D = 16t^2 - 8t - 3 = (4t-1)^2 - 4; D \leq x\} \quad \text{square-free}, \]
\[ C = \frac{1}{4} \prod_{\text{odd primes}} (1 - (2/p^2)) \quad \text{and hence} \]
\[ C > \frac{1}{4} \left( 1 - \frac{2}{9} \right) > 0 \]
holds by
\[ 1 - \frac{2}{9} = 1 - \sqrt{\frac{2}{9}} \quad \text{for any prime number} \ p \neq 3 \quad [10, 13]. \]

**Proof of Theorem 1.2.**

Let \( k \) be a real quadratic field \( Q(\tau_{L_3}) \) and \( K \) the simplest cubic fields which is defined by the cubic equation; \( x^3 = ax^2 + (a + 3)x + 1 \) with \( d_K = (a^2 + 3a + 9)^2 = d_K(\eta) \) for the field discriminant \( d_K \) and the discriminant of \( d_K(\eta) \) of a solution \( \eta \) of the equation \( x^3 - ax^2 - (a + 3)x - 1 = 0 \) derived by D. Shanks [22]. The composite field \( k \cdot K \) is denoted by \( L \). Then the field \( L \) makes a sextic bicubic extension field over the field \( Q \). Assume that \( Z_L = Z[\xi] \) for an integer \( \xi \) in \( L \). Let \( \sigma \) and \( \tau \) be generators of the Galois groups \( G(K/Q) \) and \( G(k/Q) \), respectively. Then we consider the following identity among the partial different of a number \( \xi \) in \( L \);
\[ (\xi - \xi_1^r)(\xi - \xi_2^r) - (\xi - \xi_1^r)(\xi - \xi_2^r)^r = 0. \]

Since these three products of the differents are invariant by the action \( \tau \), they belong to the the cubic field \( K \). By the assumption of \( \text{ind}_k(\xi) = 1 \), it is deduced that \( \partial_k(\xi) = \partial_L = \partial_K \alpha \) by \( \text{gcd}(\partial_K, \partial_L) = 1 \). Here \( \partial_L \) and \( \partial_M \) denote the different of a number \( \xi \) and the relative field different with respect to \( L/K \), respectively. For an ideal \( \mathfrak{c} \) and a number \( \gamma \) of a field \( M \), \( \mathfrak{c} = \gamma \) means that both ideals \( \mathfrak{c} \) and \( (\gamma) \) are equal to each other in \( M \). On the above identity, we explain the meaning for the case of a prime conductor \( p \) of \( K \).

By \( \partial_L(\xi) = (\xi - \xi_1^r)(\xi - \xi_2^r)^r \) \( (\xi - \xi_1^r)(\xi - \xi_2^r)^r = \partial_L \) it holds that \( (\xi - \xi_1^r) = (\tau_{L_3} \xi) \), \( (\xi - \xi_2^r) = \mathfrak{c} \) and \( (\xi - \xi_2^r)^r = (1) \) for the ramified prime ideals \( (\tau_{L_3}) = (\sqrt{5}) \) in \( k \) and \( \mathfrak{c} \) in \( K \) with \( (\tau_{L_3}) = (5) \) and \( \mathfrak{c}^3 = (p) \). Thus on the difference of the two products in (*) we obtain \( N_K([\xi - \xi_1^r]) (\xi - \xi_2^r)^r - (\xi - \xi_1^r)(\xi - \xi_2^r)^r = N_K([\xi - \xi_1^r]) (\xi - \xi_2^r)^r \right) = \pm 1 \), and hence \( N_K([\xi - \xi_1^r]) (\xi - \xi_2^r)^r = (\sqrt{5})^3 \) \( = \pm 1 \) \( (\text{mod} \ p) \), namely \( 5^3 + 1 = 2 \cdot 3 \cdot 7 = 0 \) or \( 5^3 - 1 = 2 \cdot 3 \cdot 31 \equiv 0 \) \( (\text{mod} \ p) \) holds. Since \( p \) is a conductor \( a^2 + 3a + 9 \) of a simplest cubic field, we obtain the simplest cubic fields \( K \), which should coincide with the maximal real subfield \( k \); for \( a = 1 \) of 7th cyclotomic \( k \), or \( k_9 \) for \( a = 0 \) of 9th cyclotomic \( k_9 \). Since a sextic field \( L \) is a relative cubic extension over the quadratic subfield \( k \), a candidate element \( \xi \) of \( Z_L = Z[\xi] \) is represented by \( \alpha + \beta \omega \) with an integer \( \alpha \), a unit \( \beta \in K \) and a unit \( \omega = (\frac{1 + \sqrt{5}}{2}) \). In fact, for the case of \( k \), we can choose \( \omega \) as \( \xi \) with the Gauß period \( \eta \) attached to a cubic character \( \psi \), and for the case of \( k_9 \) we can find \( \eta + \omega \) as \( \xi \) with the period \( \eta \) attached to a cubic character \( \psi_9 \). For an integral basis \( \{\xi_{i}\}_{i=0}^{29} \) of \( L \), we have \( \{\xi_{i}\}_{i=0}^{29} \) \( \text{mod} \ \mathbb{Z}_{2} \). The sextic field \( L \) is generated by \( \xi = \eta \omega \), which satisfies \( (\xi/\omega)^3 + (\xi/\omega)^2 - 2(\xi/\omega) - 1 = 0 \), namely by \( \xi^3 - 2\xi - 1 = (-\xi^2 + 2\xi + 2) \) it holds that \( \left( \frac{\xi^3 - 2\xi - 1}{-\xi^2 + 2\xi + 2} \right)^2 \). First we examine the fact for the sextic field \( L \) by PARI/GP, which is written in Section 4. Next since the fields \( K \) and \( k \) are linearly disjoint, that is \( K \cap k = Q \) by \( \text{gcd}(d_k, d_\lambda) = 1 \), the ring \( Z_L \) of the
composite field $L$ coincides with $Z_k \cdot Z_r = Z[\eta, \eta^2] \cdot Z[\omega] = Z[\eta, \eta^2, \omega, \eta^2 \omega, \omega^2]$. Thus for $\xi=\eta+\omega$ the representation matrix $A$ of 
$\{1, \xi, \xi^2, \xi^3, \xi^4, \xi^5\}$ with respect to $\{\eta, \eta^2, \omega, \eta^2 \omega, \omega^2\}$ is 
equal to

$$
(\{1, 1, 1, 2, 9\}, \{2, 2, 2, 15\}, \{1, 1, 1, 6, 12\},
\{2, 2, 3, 15\}, \{1, 4, 3, 25\}, \{1, 1, 2, 9, 20\}),
$$

which is equivalent to

$$
(\{\cdot, \cdot, \cdot, \cdot, \cdot\}, \{\cdot, \cdot, \cdot, \cdot, \cdot\}, \{\cdot, \cdot, \cdot, \cdot, \cdot\}, \{\cdot, \cdot, \cdot, \cdot, \cdot\}, \{\cdot, \cdot, \cdot, \cdot, \cdot\}),
$$

and hence whose determinant is equal to $-1$, namely the matrix $A$ belongs to $SL_n(Z)$, where $\cdot$ means $0$ and 
$\cdot M$ for a matrix $M$ denotes the transposed one. Thus the sctic field 
$L=k \cdot k^*$ is actually monogenic.

In the case of $L=k \cdot k^*$, the choice $\xi=\eta+\omega$ would be 
failed, where the Gauß period $\eta$ is a root of 
g($y$) = $y^3 - 3y + 1$. Then we select $\eta+\omega$ as a candidate 
of a power integral basis; $Z[\xi]=Z_L$. Since the simplest cubic field is monogenic, 
$N_L((\xi-\xi^2))(\xi-\xi^2)$ 
$= N_L((\eta-\eta^2)(\eta-\eta^2)) = p^2$ holds. Thus it follows that 
$N_L(d_L) = (\xi-\xi^2) = p^2$ and 
$N_L(d_L(\xi-\xi^2)) = 5^3$. On the other hand, by 
$\delta_k = \delta_k e_k$ it is deduced that 
$d_L = N_L(\delta_k) N_L(\delta_k) = d_L^{d(\xi)} = (3^2)^2 \cdot 5^3 = 3^4 \cdot 5^3 = 820125$. Here for an 
ideal $\mathfrak{p}$ in a field $M$, $N_L(\mathfrak{p})$ means the ideal norm of 
$\mathfrak{p}$ with respect to $M$. 
Then we must confirm that the partial factor $\xi-\xi^2$ and hence $\xi-\xi^2 r$ 
are not obstacle factors, namely they are units in $L$. We take the relative norm 
$N_L((\xi-\xi^2)) = N_L((\eta-\eta^2)+(\xi-\xi^2)) = (\eta-\eta^2) + (\xi-\xi^2) + (\eta-\eta^2) \cdot (\xi-\xi^2) + (\eta-\eta^2)+(\xi-\xi^2) \cdot (\eta-\eta^2) + (\eta-\eta^2) \cdot (\eta-\eta^2) \cdot 5 + 5^3 = 5^3$. On the first 
product, we obtain $C = 3 N_L(\eta)$ and 
$D = (\eta+\eta^2)+(\eta+\eta^2)+(\eta+\eta^2)$. By 
$(\eta+\eta^2)+(\eta+\eta^2)+(\eta+\eta^2)$, it follows that 
$C + D = 3 N_L(\eta)$. We obtain 
$C = D = B_1 + 3 N_L(\eta)$ 
$+ S_1 N_L(\eta)$. Here we use the relations 
$B_1 = B_3 + (D + C) N_L(\eta)$ with 
$B_3 = (\eta+\eta^2)+(\eta+\eta^2)+(\eta+\eta^2)$, 
$B_3 = (\eta+\eta^2)+(\eta+\eta^2)+(\eta+\eta^2)$. Then we have 
$B_3 = 24$ and $S_3 = -3$, and hence $C \cdot D = -18$. Thus the set 
$\{C, D\}$ of values is equal to $\{-6, 3\}$. Then it deduces for the derivative 
g($y$) of $g(y)$ that 
$N_L(\xi^{\xi^3}) = -c + D + (\eta+\eta^2) - (\eta+\eta^2) + (\eta+\eta^2) \cdot 5 + 5^3 = 9 - 4 \cdot 5$, and hence 
$N_L(\xi^{\xi^3}) = 81 - 16 \cdot 5 = 1$.

3. EXPERIMENTS AND FUTURE WORKS

To find new phenomena on Number Theory, experiments by PARI/GP are sometimes indispensable. Let $L$ be the cyclic sextic field $Q(\eta, \omega)$ over the simplest cubic field with a root $\eta$ of the cubic polynomial 
$x^3 = ax^2 + (a+3)x + 1$ and a unit

$$
\omega = \frac{1 + \sqrt{5}}{2}
$$
in the real quadratic field with prime discriminant $5$. Select a number $\eta+\omega$ as a candidate of integral power basis; $Z_L = Z[\xi] = Z[\{1, \overline{\xi}, \overline{\xi}^2\}]$.

PARI/GP gives an affirmative answer as follows.

\fbox{\text{Then PARI/GP gives a power integral basis}} 
$g \rightarrow$ nbasis((x^3-8*x-1)^2-(x^3-2*x+1)*(x^2+2*x+2)-(-x^2+2*x+2)^2) \#the field discriminant of $d_L$ (of the sextic field L 
gp> \text{nfdis}(x^3-3*x-1)^2-(x^3-2*x+1)^2 \#and the prime number decomposition of $d_L$ (of $p$-field) \text{factor}(300125) \#namely} 
\text{d_L[5]=5^3 \cdot d_{-1}[5]} \text{\#d_{-1}[5]} = 5^3 \cdot \text{d_{-1}[5]} \text{\#d_{-1}[5]=5^3 \cdot d_{-1}[5]} = 5^3 \cdot d_{-1}[5] = 5^3 \cdot d_{-1}[5] = 5^3 \cdot d_{-1}[5].$

Since the fields $Q(\tau_{\delta}) = Q(\sqrt{5})$ and the simplest cubic field $Q(\eta)$ with $\eta = -\eta^2 + 2 + \eta + 1$ are linearly disjont, that is, $(\partial Q(\tau_{\delta}))\partial Q(\eta)) = 1$, the set 
$\{\eta \cdot \omega\} \not\equiv 0 \mod{5}$ makes an integral basis of $L$. Let $A$ be the representation matrix of 
$\{\xi \cdot \omega\} \not\equiv 0 \mod{5}$, then it turns out that $A$ belongs to 
$SL_n(Z)$ in Section 3. Then for $\xi=\eta+\omega$ it is deduced that 
$Z[\xi] = Z_L$, namely the experiment is correct.

FUTURE WORKS

1. Generalize Thorem 1.2 for the cyclic sextic fields $L = K \cdot k$ in which any simplest cubic field $K$ and any 
real or imaginary quadratic field $k$ with $(\delta_k, \delta_k) = 1$.

2. Let $p$ and $\xi_p$ be a prime number and $p$ the prime number of $p$-field, respectively and $F_p$ the finite field of 
$p$ element. Let $\tau_{\chi}$ be the Gauß sum $\sum_{x \in F_p} \chi(x) \xi^x_{p}$ attached to the non-trivial character $\chi$ belonging to the
character group \( \hat{F}_p^\times \) with the multiplicative group \( F_p^\times = F_p \setminus \{0\} \). Let \( J(\chi, \lambda) = \sum_{x,y \in F_p} \chi(x) \lambda(y) \) be the Jacobi sum attached to the non-trivial characters \( \chi \) and \( \lambda \). Then the relation

\[
J(\chi, \lambda) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}
\]

of Gauß sum and Jacob sum is deduced [12]. Let \( \Gamma(x), B(x, y) \) be the Gamma function

\[
\int_0^1 e^{-t^{-i} \sigma} \, dt \quad (\Re(x) > 0)
\]

and Beta function \( \int_0^1 t^{-i}(1-t)^{-i} \, dt \quad \Re(x), \Re(y) > 0 \), respectively. Then the next iteration is well known;

\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}.
\]

Thus find a suitable interpretation between Jacobi sum and Beta function.

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