Estimating the Population Standard Deviation with Confidence Interval: A Simulation Study under Skewed and Symmetric Conditions

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Abstract: This paper investigates the performance of ten methods for constructing a confidence interval estimator for the population standard deviation by a simulation study. Since a theoretical comparison among the interval estimators is not possible, a simulation study has been conducted to compare the performance of the selected interval estimators. Data were randomly generated from several distributions with a range of sample sizes. Various evaluation criterions are considered for performance comparison. Two health related data have been analyzed to illustrate the application of the proposed confidence intervals. Based on simulation results, some intervals with the best performance have been recommended for practitioners.

Keywords: Bootstrapping, Coverage probability, Interval estimator, Kurtosis, Robustness, Scale estimator, Skewed Distribution.

1. INTRODUCTION

A term often is used in the Inferential Statistics is known as confidence interval (CI). It measures the chance that a value will fall between a lower and an upper bound of a probability distribution. Given a 95% CI, say for an example, a stock XYZ's return will fall between 5.3% and 10.5% over the next year. It means that we are 95% confident that the return of holding XYZ stock over the next year will fall between 5.3% and 10.5%. It is well-known that the most commonly used scale estimator, namely sample standard deviation (S) is very important in many statistical applications which provides a logical point estimate of the population standard deviation (σ). The classical chi-square 100(1-α)% CI for σ is based on the assumption that the underlying distribution of data is normal with no outliers. Unfortunately, this assumption for constructing CI for σ is very sensitive to normality assumption [1, 2]. Many research papers in literature shows that S is not an efficient scale estimator in skewed and/or leptokurtic distributions. As a result, it is not robust to slight deviations from normality, see [3] for details. Thus, when the parent population is far from normal, achieved confidence level of CI of σ is far from the nominal level. Despite S being the most efficient scale estimator for normal distribution and often used to construct the $100(1-\alpha)\%$ CI for σ but the basic question is what happens if the data are not from a normal distribution but instead from heavier tails or from skewed distributions. Thus, the need for alternatives to the classical chi-square $100(1-\alpha)\%$ CI for σ comes to play. Although much work has been done on improving CI for population mean [4-7], improving robustness for CI for σ is still greatly unexplored.

The aim of this paper is (i) to evaluate and compare several available methods for constructing CI for σ and (ii) based on extensive simulation and numerical examples to seek evidences to recommend some CI with best performance for researchers. Performances of the proposed methods are investigated through a Monte Carlo simulation study based on some evaluation criteria such as coverage probability, average width and SD of width. The coverage probability naturally varies from distribution to distribution for a given procedure, but a good procedure should keep this variation small. Furthermore, we want a CI whose endpoints are generally close together, thus a small average width is good [8]. Two health related data are used to illustrate the results given in the paper.

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The organization of the paper is as follows. In section 2, different alternative methods for constructing CI for σ are presented. A Monte Carlo simulation study with flowchart is outlined in section 3. To illustrate findings of the paper, some real life data are analyzed in section 4. Finally, some concluding remarks and future research are presented in section 5.

2. METHODS FOR CONSTRUCTING CI FOR σ

Suppose that $x_1, x_2, \ldots, x_n$ is a random sample of size n from the normal distribution, i.e. $X_i \sim N(\mu, \sigma^2)$ for all *i*, then $\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \chi^2_{n-1}$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the sample variance.

2.1. Exact CI

The classical chi-square $100(1-\alpha)\%$ CI for σ is given as follows:

$$LCL = \sqrt{\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2},(n-1)}}} \quad and \quad UCL = \sqrt{\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2},(n-1)}}} \qquad (1)$$

Where $\chi^2_{\frac{\alpha}{2}}$ and $\chi^2_{1-\frac{\alpha}{2}}$ are the $(\alpha/2)^{th}$ and $(1-(\alpha/2))^{th}$ percentile points of the χ^2 distribution with (n-1) df.

2.2. Robust CI

The exact CI for σ in (1) is sensitive to minor violations of the normality assumption. Bonett [9] show that (1) has an asymptotic coverage probability of about 76, 63, 60 and 51 for the Logistic, the Student t(7), the Laplace and the Student t(5) distributions respectively. These results are not very acceptable as symmetric distributions are not easily distinguishable from a normal distribution unless n is large. Also, the exact CI for σ in (1) as demonstrated by Lehman [10] is highly sensitive to the presence of outliers and/or to departure from normality.

Following above, [2] proposed an alternative to the exact $100(1-\alpha)\%$ CI for σ based on [11] estimator, Q_n [2]. A brief description of this method is given below:

Instead of assuming $X_i \sim N(\mu, \sigma^2)$, let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a continuous, independent and identically distributed random variable. The $100(1-\alpha)\%$ robust CI for σ is as follows:

$$LCL = \frac{DQ_n}{Z_{\frac{\alpha}{2}} + D_1} \quad and \quad UCL = \frac{DQ_n}{Z_{1-\frac{\alpha}{2}} + D_1} \qquad (2)$$

where $D = 1.28\sqrt{n} * d_n$, An approximation result of D for larger values of n can be calculated as follows:

$$D = \begin{cases} (1.28\sqrt{n})(\frac{n}{n+1.4}) & for\ odd\ values\ of\ n \\ (1.28\sqrt{n})(\frac{n}{n+3.8}) & for\ even\ values\ of\ n \end{cases}$$

$D_1 = 1.28\sqrt{n}$, Q_n is the [11] estimator defined as $Qn = 2.2219\left\{|X_i - X_j|;\ i<j\ ;\ i = 1,2,3,\ldots,n\ ;\ j = 1,2,3,\ldots,n\right\}_{\{g\}}$, and $g = \binom{h}{2} \approx (\binom{n}{2}/4)$ where $h = \left(\left[\frac{n}{2}\right]+1\right)$ (i.e., roughly half the number of observations). Here the symbol (.) represents combination and symbol [.] is used to take only integer part of a fraction. Q_n estimator is the *g-th* order statistic of $\binom{n}{2}$ inter-point distances. The value 2.2219 is chosen to make Q_n a consistent estimator of scale for normal data (For values of d_n see [2].

2.3. Bonett CI

Let $X_i \sim N(\mu, \sigma^2)$ for all *i*. [9] proposed the following (1-α)100% CI for σ:

LCL = sqrt($\exp\left\{\ln(c\hat{\sigma}^2) - Z_{\alpha/2}se\right\}$) and UCL = sqrt($\exp\left\{\ln(c\hat{\sigma}^2) + Z_{\alpha/2}se\right\}$), (3)

where $Z_{\alpha/2}$ is two-sided critical z-value, $se = c[\{\hat{\gamma}_4(n-3)/n\}/(n-1)]^{1/2}$, c = $n/(n-Z_{\alpha/2})$ and $\hat{\gamma}_4 = n\sum_{i=1}^{n}(Y_i - \hat{\mu})^4 / (\sum(Y_i - \hat{\mu})^2)^2$.

2.4. Steve Large Sample Normal Approximations CI

Steve [12] proposed the following $100(1-\alpha)\%$ CI for σ:

$$LCL = \sqrt{\frac{S^2}{1 - Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{\gamma}-1}{n}}}} \quad and \quad UCL = \sqrt{\frac{S^2}{1 + Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{\gamma}-1}{n}}}} \qquad (4)$$

where $\hat{\gamma}$ is the kurtosis estimator.

2.5. Hummel, Banga and Hettmansperger CIs

2.5.1. Log asymptotic Approximation (Ln Asympt) CI

Because of high skewness of the distribution S^2 for small n, [1] hereafter HBH, using the method of applying natural log to S^2 in (4) (i.e. by adjusting skewness) proposed the following $100(1-\alpha)\%$ CI for σ:

$$LCL = sqrt(s^2 \exp\left(Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{\gamma}-1}{n}}\right)) \text{ and}$$

$$UCL = sqrt(s^2 \exp\left(-Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{\gamma}-1}{n}}\right)) \qquad (5)$$

2.5.2. Adjusted Degrees of Freedom (AdjDF) CI

To find a better CI, HBH adjusted DF (i.e. (n-1)) of the exact CI for σ in (1) and proposed the following $100(1-\alpha)\%$ CI for σ:

$$LCL = \sqrt{\frac{\hat{r}S^2}{\chi^2_{\frac{\alpha}{2},\hat{r}}}} \text{ and } UCL = \sqrt{\frac{\hat{r}S^2}{\chi^2_{1-\frac{\alpha}{2},\hat{r}}}} \qquad (6)$$

where $\hat{r} = \dfrac{2n}{\hat{\gamma}e + (\frac{2n}{n-1})}$ and $\hat{\gamma}e$ is the estimate of kurtosis excess (e.g. for normal distribution $\hat{\gamma}e = \hat{\gamma} - 3 = 0$ because $\hat{\gamma}$ =3 for normal distribution) which is defined as

$$\hat{\gamma}e = \frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum_{i=1}^{n}\frac{(x_i-\overline{x})^4}{S^4} - \frac{3(n-1)^2}{(n-2)(n-3)} \qquad (7)$$

If a random sample comes from the normal population, then r = n-1 and (6) reduces to (1).

2.5.3. Modified Adjusted Degrees of Freedom (B-C AdjDF) CI

Due to low coverage probabilities of UCL of the AdjDF method (see equation 6), HBH adjusted only 100(1-α)% UCL in the following way:

$$UCL_{B-C} = sqrt(S^2\left(\frac{2\hat{r}}{\chi^2_{1-\alpha,\hat{r}}} + C_{1-\alpha}\left(\frac{n+1}{(n-1)(2+\hat{r})}\right) - 1\right)) \qquad (8)$$

where

$$C_{\alpha(r)} = \sqrt{2}Z_\alpha\sqrt{r} + \frac{2}{3}(Z_\alpha^2 - 1)r + \frac{1}{9\sqrt{2}}(Z_\alpha^3 - 7Z_\alpha)r^{\frac{3}{2}}$$
$$-\frac{1}{405}(6Z_\alpha^4 + 14Z_\alpha^2 - 433)r^2 + \frac{1}{4860\sqrt{2}}(9Z_\alpha^5 + 256Z_\alpha^3 - 433Z_\alpha)r^{\frac{5}{2}}$$

and other terms are defined as above.

2.6. Bootstrap CIs

Let $X^{(*)} = X_1^{(*)}, X_2^{(*)}, \ldots, X_n^{(*)}$, where the i^{th} sample is denoted by $X^{(i)}$ for i=1,2 ,…, B and B is the number of bootstrap samples. Compute σ for all bootstrap samples.

2.6.1. Proposed Non-Parametric Bootstrap CI

Order the SDs of all bootstrap samples as follows: $S^*_{(1)} \le S^*_{(2)} \le S^*_{(3)} \ldots \le S^*_{(B)}$.

CI for population σ:

$$LCL = S^*_{[(\alpha/2)B]} \text{ and } UCL = S^*_{[(1-(\alpha/2))B]} \qquad (9)$$

2.6.2. Proposed Parametric Bootstrap χ^2 CI

CI for population σ:

$$\text{LCL} = S\sqrt{(n-1)/\chi^{*2}_{\alpha/2,(n-1)}} \text{ and}$$
$$\text{UCL} = S\sqrt{(n-1)/\chi^{*2}_{1-(\alpha/2),(n-1)}} \qquad (10)$$

where $\chi^{*2}_{\alpha/2}$ and $\chi^{*2}_{1-(\alpha/2)}$ are *(α/2)*th and (1-*(α/2))*th sample quintiles of $\chi^2 = \dfrac{(n-1)S^2}{\hat{\sigma}_B}$,

$\hat{\sigma}_B = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}(\overline{x}_i^* - \overline{\overline{x}})^2}$ is the bootstrap SD, $\overline{x}_i^*$ is the i^{th} bootstrap sample mean and $\overline{\overline{x}}$ is the bootstrap mean.

2.6.3. Cojbasic and Tomovic (CT) CI

Based on t-statistic, [13] proposed the following nonparametric bootstrap t CI for σ:

$$I_{boot} = S^2 - \hat{t}^{(\alpha)}\sqrt{\hat{\text{var}}(S^2)} \qquad (11)$$

where S^2 is the sample variance, $\hat{t}^{(\alpha)}$ is the α percentile of T* defined as $T^* = \dfrac{S^{2*} - S^2}{\sqrt{\hat{\text{var}}(S^{2*})}}$, S^{2*} is a bootstrap replication of statistic S^2 and $\hat{\text{var}}(S^2)$ is a consistent estimator of S^2, defined by $2\sigma^4/(n-1)$.

3. SIMULATION STUDY

Since, a theoretical comparison among selected intervals is not possible, a simulation study has been conducted to compare performances of the selected intervals. The simulation plan and discussion of the results are given in this section.

3.1. Simulation Plan

MATLAB [14] programming language was used to run the simulations and to make necessary tables.

The flowchart of our simulation is as follows:

We used random samples of sizes n = 15, 25, 50 and 100. Random samples are generated from various symmetric (light and heavy tailed) and skewed distributions which are:

(a) Standard normal distribution N(0,1) with skewness zero.

(b) Gamma(3,1) distribution with skewness 2.

(c) t-distribution with 8 df with skewness 2.

(d) Beta(5,1) distribution with skewness -1.8232.

(e) Lognormal distribution with mean 3 and SD 0.75 with skewness 3.2629.

(f) Chi-square distribution with 7 df with skewness 1.0690.

We used 5000 replications and 1500 bootstrap samples for each sample size n. The most common 95% CI (α=0.05) for confidence coefficient is used. It is well known that if data are from a symmetric distribution (or n is large), coverage probability will be exact or close to (1-α). So coverage probability is a useful criterion for evaluating CI. Another criterion is the width of CI. A shorter width (tighter CI) gives a better CI. It is obvious that when coverage probability is the same, a smaller width indicates that method is appropriate for the specific sample. In order to compare performance of various intervals, the following criteria are considered: (i) Coverage probabilities (below, coverage, and above), (ii) Mean width and (iii) SD of width. Below (above) rate of a CI is the fraction out of 5000 samples that resulted in an interval that lies entirely above (below) the true value of σ. The coverage probability is found as sum of lower rate and upper rate and then subtracted from total probability of 1. Simulation results are reported in Tables **1-6** respectively for different selected distributions and for better understanding graphical methods are selected.

3.2. Simulation Results and Discussions

In Tables **1-6**, we have tabulated performances of the selected intervals for standard normal, gamma, t, beta, lognormal and chi-square distributions. For better understanding, the cover rate is presented in Figures **1-6**.

Table **1** and the Figure **1** present the simulation results for the normal distribution case. It is not a surprise to see the ordinary chi square interval to perform the best with very close coverage to the nominal value across all sample sizes selected. Nonparametric confidence interval seems to have the second best performance in the normally distributed data. In general, it is apparent that coverage probability increases as the sample size increases. On the other hand, the AdjDF confidence interval seems to underestimate the coverage probability for all sample sizes in the normally distributed data.

Table **2** and Figure **2** present the simulation results for the Gamma distribution case. The robust, nonparametric and the parametric bootstrap confidence intervals seem to perform better especially for large sample size (n=100). The classical chi-square, the AdjDF, the B-C and the CT nonparametric bootstrap seems to underestimate the coverage probability for data from the Gamma distribution.

Table **3** and Figure **3** present the simulation results for the t distribution with 8 degrees of freedom. For sample size at least 50, Bonett and nonparametric confidence interval seems to have better performance compared to others, with the AdjDF and B-C still underestimating the coverage probability. In Table **4** and Figure **4**, we have presented simulation results for the Beta (5,1) distribution. In the presence of the negatively skewed data, we observe that for sample sizes of at least 50, Bonett, Steve, Ln_A, non-parametric and parametric bootstrap intervals have confidence cover rate close to the nominal level. We see that with increasing sample sizes, confidence probability close to the nominal level. The AdjDF and B-C intervals cover rate found below the nominal level.

Table **5** and the Figure **5** report the simulation results for the lognormal (3,0.75) distribution in the case of highly skewed data. It is easy to see that in the presence of highly skewed data, the robust, the

Table 1: Coverage Probabilities for N(0,1) Distribution with Skewness 0

	Selected CIs									
Criterion	**O_Chi**	**Robust**	**Bonett**	**Steve**	**Ln_A**	**AdjDF**	**B-C**	**N_Boot**	**P_Boot**	**CT_B**
n=15										
Below rate	0.0264	0.0200	0.1208	0.0376	0.0348	0.1092	0.1092	0.0284	0.0088	0.1160
Cover rate	0.9480	0.8412	0.8724	0.8928	0.8672	0.7168	0.7972	0.9580	0.7688	0.8840
Over rate	0.0256	0.1388	0.0068	0.0696	0.0980	0.1740	0.0936	0.0136	0.2224	0.0000
Mean width	0.8329	1.0382	0.6297	1.0785	0.6340	0.8298	0.9000	0.6305	0.7517	1.4321
SD width	0.1581	0.2450	0.1544	1.9774	0.1979	0.3386	0.3813	0.1742	0.1427	0.5347
n=25										
Below rate	0.0260	0.0064	0.0664	0.0244	0.0264	0.0812	0.0920	0.0236	0.0208	0.0744
Cover rate	0.9488	0.8592	0.9280	0.9228	0.9096	0.7304	0.8084	0.9560	0.7900	0.9244
Over rate	0.0252	0.1344	0.0056	0.0528	0.0640	0.1884	0.0996	0.0204	0.1892	0.0012
Mean width	0.6069	0.7224	0.5563	0.6435	0.5122	0.5956	0.6323	0.5087	0.5010	1.1309
SD width	0.0864	0.1262	0.1081	0.3150	0.1282	0.1748	0.1985	0.1171	0.0713	0.3195
n=50										
Below rate	0.0256	0.0164	0.0436	0.0276	0.0252	0.0920	0.0812	0.0192	0.0304	0.0736
Cover rate	0.9492	0.8796	0.9528	0.9256	0.9192	0.7324	0.8156	0.9648	0.8980	0.9244
Over rate	0.0252	0.1040	0.0036	0.0468	0.0556	0.1756	0.1032	0.0160	0.0716	0.0020
Mean width	0.4076	0.4883	0.4311	0.4102	0.3727	0.4004	0.4111	0.3740	0.3869	0.7858
SD width	0.0418	0.0568	0.0625	0.0893	0.0693	0.0794	0.0872	0.0650	0.0396	0.1608
n=100										
Below rate	0.0212	0.0084	0.0256	0.0248	0.0232	0.0616	0.0616	0.0260	0.0912	0.0504
Cover rate	0.9548	0.9080	0.9708	0.9352	0.9360	0.7660	0.8352	0.9524	0.9028	0.9468
Over rate	0.0240	0.0836	0.0036	0.0400	0.0408	0.1724	0.1032	0.0216	0.0060	0.0028
Mean width	0.2829	0.3254	0.3225	0.2839	0.2710	0.2808	0.2802	0.2700	0.2506	0.5698
SD width	0.0199	0.0257	0.0336	0.0413	0.0366	0.0391	0.0413	0.0364	0.0176	0.0798

Notes: O_Chi: Ordinary Chi-Square given in (1), Ln_A: Ln Asympt given in (6), B-C: B-C AdjDF given (8), N_boot: Non-parametric Bootstrap given (9), P_Boot: Parametric Bootstrap given in (10) and CT_B: Cojbasic and Tomovic Bootstrap given (11).

Table 2: Coverage Probabilities for Gamma (3,1) Distribution with Skewness 2

	Selected CIs									
Criterion	**O_Chi**	**Robust**	**Bonett**	**Steve**	**Ln_A**	**AdjDF**	**B-C**	**N_Boot**	**P_Boot**	**CT_B**
n=15										
Below rate	0.1768	0.4680	0.2820	0.1144	0.0432	0.0244	0.0244	0.0180	0.0016	0.1520
Cover rate	0.6684	0.5312	0.6848	0.7892	0.7232	0.6092	0.6420	0.9596	0.5564	0.3816
Over rate	0.1548	0.0116	0.0332	0.0964	0.2336	0.3664	0.3336	0.0224	0.4420	0.4664
Mean width	2.4183	2.0736	2.3142	5.6337	2.5880	4.4530	4.7586	2.3897	7.4903	12.1519
SD width	0.8442	0.6773	1.3126	11.0832	1.7162	4.8174	4.8161	1.3158	2.6147	9.4641
n=25										
Below rate	0.1768	0.3184	0.2028	0.0980	0.0296	0.0024	0.0024	0.0224	0.0116	0.1288
Cover rate	0.7008	0.6700	0.7780	0.8208	0.7860	0.6156	0.6428	0.9376	0.7140	0.5988
Over rate	0.1224	0.0008	0.0192	0.0812	0.1844	0.3820	0.3548	0.0400	0.2744	0.2724
Mean width	1.7816	1.4084	2.2379	4.8849	2.3184	3.2109	3.4970	2.2018	3.6328	9.3015
SD width	0.4844	0.3415	1.1357	8.6046	1.3475	2.6915	2.8628	1.1391	0.9877	5.3407
n=50										
Below rate	0.1616	0.2212	0.1388	0.0808	0.0200	0.0000	0.0000	0.0288	0.0660	0.3916
Cover rate	0.7212	0.7788	0.8504	0.8624	0.8388	0.6304	0.6524	0.9308	0.8760	0.6024
Over rate	0.1172	0.0000	0.0108	0.0568	0.1412	0.3696	0.3476	0.0404	0.0580	0.0060
Mean width	1.2041	0.9492	1.8702	3.3232	1.8320	2.1591	2.3344	1.7957	1.8317	7.0344
SD width	0.2346	0.1596	0.7854	5.9067	0.8649	1.2837	1.4334	0.7681	0.3569	2.8281
n=100										
Below rate	0.1496	0.0188	0.0996	0.0652	0.0172	0.0000	0.0000	0.0268	0.0196	0.3112
Cover rate	0.7452	0.9812	0.8904	0.8864	0.8760	0.6860	0.7036	0.9436	0.9580	0.6608
Over rate	0.1052	0.0000	0.0100	0.0484	0.1068	0.3140	0.2964	0.0296	0.0224	0.0280
Mean width	0.8440	0.6297	1.5048	1.8336	1.4417	1.5695	1.6593	1.3960	1.3418	4.9846
SD width	0.1199	0.0738	0.5275	1.3004	0.5606	0.7000	0.7901	0.4941	0.1907	1.4385

Notes: See Table **1**.

Table 3: Coverage Probabilities for t Distribution with 8 df and Skewness 2

	Selected CIs									
Criterion	**O_Chi**	**Robust**	**Bonett**	**Steve**	**Ln_A**	**AdjDF**	**B-C**	**N_Boot**	**P_Boot**	**CT_B**
n=15										
Below rate	0.0604	0.0380	0.1796	0.0544	0.0296	0.0040	0.0040	0.0256	0.0340	0.0556
Cover rate	0.8736	0.8724	0.8108	0.8752	0.8316	0.7440	0.8024	0.9436	0.5816	0.5892
Over rate	0.0660	0.0896	0.0096	0.0704	0.1388	0.2520	0.1936	0.0308	0.3844	0.3552
Mean width	0.9472	1.1172	0.7719	1.6057	0.8095	1.1377	1.2388	0.7695	0.7509	1.9770
SD width	0.2237	0.2884	0.2868	2.2909	0.3718	0.7900	0.8440	0.2974	0.1774	0.9815
n=25										
Below rate	0.0516	0.0488	0.1096	0.0468	0.0228	0.0096	0.0096	0.0288	0.0072	0.1112
Cover rate	0.8844	0.8876	0.8820	0.8940	0.8728	0.7196	0.7800	0.9524	0.8656	0.8488
Over rate	0.0640	0.0636	0.0084	0.0592	0.1044	0.2708	0.2104	0.0188	0.1272	0.0400
Mean width	0.6935	0.7807	0.7091	1.1304	0.6875	0.8385	0.9046	0.6511	0.8742	1.4628
SD width	0.1257	0.1509	0.2298	3.0172	0.2735	0.4204	0.4736	0.2150	0.1584	0.5432
n=50										
Below rate	0.0496	0.0376	0.0628	0.0384	0.0200	0.0272	0.0272	0.0200	0.4380	0.4644
Cover rate	0.8928	0.9008	0.9288	0.9120	0.9072	0.6488	0.7088	0.9508	0.5568	0.5352
Over rate	0.0576	0.0616	0.0084	0.0496	0.0728	0.3240	0.2640	0.0292	0.0052	0.0004
Mean width	0.4726	0.5315	0.5754	0.6401	0.5276	0.5842	0.6133	0.6466	0.3665	1.0958
SD width	0.0640	0.0689	0.1908	0.4586	0.2121	0.3110	0.3413	0.2206	0.0497	0.3309
n=100										
Below rate	0.0488	0.0724	0.0468	0.0376	0.0212	0.0572	0.0572	0.0232	0.0456	0.1204
Cover rate	0.8976	0.9028	0.9472	0.9232	0.9136	0.5928	0.6556	0.9508	0.8992	0.8580
Over rate	0.0536	0.0248	0.0060	0.0392	0.0652	0.3500	0.2872	0.0260	0.0552	0.0216
Mean width	0.3260	0.3529	0.4318	0.4238	0.3855	0.4047	0.4122	0.3678	0.3505	0.7504
SD width	0.0302	0.0317	0.1023	0.2323	0.1111	0.1277	0.1419	0.0936	0.0325	0.1412

Notes: See Table **1**.

Table 4: Coverage Probabilities for Beta(5,1) Distribution with Skewness -1.8232

	Selected CIs									
Criterion	**O_Chi**	**Robust**	**Bonett**	**Steve**	**Ln_A**	**AdjDF**	**B-C**	**N_Boot**	**P_Boot**	**CT_B**
n=15										
Below rate	0.0740	0.4108	0.2000	0.0832	0.0412	0.1060	0.1060	0.0284	0.0200	0.2176
Cover rate	0.8716	0.5892	0.7864	0.8336	0.7920	0.6488	0.7236	0.9432	0.6196	0.7824
Over rate	0.0544	0.0000	0.0136	0.0832	0.1668	0.2452	0.1704	0.0284	0.3604	0.0000
Mean width	0.1156	0.1171	0.0956	0.2150	0.1006	0.1453	0.0505	0.0996	0.1364	0.0281
SD width	0.0280	0.0327	0.0343	0.3474	0.0454	0.0980	0.0109	0.0396	0.0330	0.0135
n=25										
Below rate	0.0696	0.2652	0.1200	0.0544	0.0328	0.0908	0.0908	0.0220	0.0392	0.1968
Cover rate	0.8724	0.7312	0.8672	0.8724	0.8532	0.6716	0.7356	0.9504	0.8932	0.8032
Over rate	0.0580	0.0036	0.0128	0.0732	0.1140	0.2376	0.1736	0.0276	0.0676	0.0000
Mean width	0.0848	0.0817	0.0881	0.1462	0.0857	0.1054	0.1139	0.0841	0.1425	0.0202
SD width	0.0155	0.0176	0.0263	0.2009	0.0319	0.0486	0.0550	0.0281	0.0260	0.0073
n=50										
Below rate	0.0648	0.1872	0.0672	0.0448	0.0224	0.0500	0.0500	0.0152	0.0000	0.1668
Cover rate	0.8856	0.7988	0.9260	0.9056	0.8992	0.6904	0.7428	0.9632	0.9060	0.8284
Over rate	0.0496	0.0140	0.0068	0.0496	0.0784	0.2596	0.2072	0.0216	0.0940	0.0048
Mean width	0.0574	0.0554	0.0705	0.0779	0.0649	0.0715	0.0752	0.0640	0.0623	0.0151
SD width	0.0072	0.0080	0.0156	0.0355	0.0176	0.0215	0.0243	0.0165	0.0078	0.0038
n=100										
Below rate	0.0604	0.1160	0.0372	0.0272	0.0212	0.0280	0.0280	0.0172	0.0072	0.1152
Cover rate	0.8832	0.8548	0.9560	0.9312	0.9268	0.7108	0.7636	0.9656	0.9760	0.8708
Over rate	0.0564	0.0292	0.0529	0.0416	0.0520	0.2612	0.2084	0.0172	0.0168	0.0140
Mean width	0.0398	0.0367	0.0083	0.0511	0.0473	0.0496	0.0453	0.0470	0.0592	0.0110
SD width	0.0035	0.0037	0.0038	0.0112	0.0091	0.0100	0.0123	0.0088	0.0053	0.0020

Notes: See Table **1**.

Table 5: Coverage Probabilities for Lognormal (3,0.75) Distribution with Skewness 3.2629

	Selected CIs									
Criterion	**O_Chi**	**Robust**	**Bonett**	**Steve**	**Ln_A**	**AdjDF**	**B-C**	**N_Boot**	**P_Boot**	**CT_B**
n=15										
Below rate	0.3172	0.4216	0.3736	0.1628	0.0368	0.0000	0.0000	0.0228	0.0156	0.4196
Cover rate	0.4912	0.5736	0.5952	0.7452	0.6444	0.5772	0.6080	0.9608	0.5132	0.4640
Over rate	0.1916	0.0048	0.0312	0.0920	0.3188	0.4228	0.3920	0.0164	0.4712	0.1164
Mean width	17.7645	14.5656	18.2198	40.2524	20.8625	38.9665	41.1572	19.3970	90.2271	704.1570
SD width	7.8449	4.6017	13.1559	65.5073	17.1091	50.7715	49.9112	13.8067	39.8449	821.7863
n=25										
Below rate	0.3024	0.3176	0.2964	0.1528	0.0324	0.0000	0.0000	0.0204	0.0000	0.3776
Cover rate	0.5364	0.6820	0.6728	0.7660	0.6952	0.5792	0.6328	0.9480	0.5956	0.5108
Over rate	0.1612	0.0004	0.0308	0.0812	0.2724	0.4208	0.3672	0.0316	0.4044	0.1116
Mean width	13.2422	9.9536	18.7886	40.5456	20.0204	31.4447	33.7228	18.3004	34.6845	596.1258
SD width	5.0325	2.2631	13.5612	62.0221	16.0029	41.2178	41.0870	12.6592	13.1812	587.3846
n=50										
Below rate	0.2764	0.0768	0.2260	0.1272	0.0164	0.0000	0.0000	0.0204	0.0004	0.4704
Cover rate	0.5876	0.9232	0.7596	0.8016	0.7628	0.6448	0.6396	0.9576	0.6632	0.5128
Over rate	0.1360	0.0000	0.0144	0.0712	0.2208	0.3552	0.3604	0.0220	0.3364	0.0168
Mean width	9.1533	6.6811	17.3491	34.3152	17.5468	23.0647	25.0409	16.1378	14.3138	433.2265
SD width	2.6139	1.0451	11.4679	59.6452	12.5365	24.3287	25.5348	9.8277	4.0876	284.9240
n=100										
Below rate	0.2444	0.0008	0.1716	0.1040	0.0156	0.0000	0.0000	0.0296	0.0176	0.4296
Cover rate	0.6468	0.9592	0.8164	0.8404	0.8072	0.6756	0.6416	0.9468	0.9368	0.5220
Over rate	0.1088	0.0400	0.0120	0.0556	0.1772	0.3244	0.3584	0.0236	0.0456	0.0484
Mean width	6.4167	4.4301	14.6205	26.8553	14.4405	16.9179	18.2981	13.4852	17.7357	287.7666
SD width	1.3627	0.4745	8.3297	70.4268	8.7546	13.3320	14.7615	7.1230	3.7666	135.2810

Notes: See Table **1**.

Table 6: Coverage Probabilities for Chi-Square Distribution with 7 df and Skewness 1.0690

	Selected CIs									
Criterion	**O_Chi**	**Robust**	**Bonett**	**Steve**	**Ln_A**	**AdjDF**	**B-C**	**N_Boot**	**P_Boot**	**CT_B**
n=15										
Below rate	0.0724	0.1528	0.1840	0.0648	0.0360	0.0764	0.0764	0.0264	0.1532	0.1488
Cover rate	0.8508	0.8444	0.8024	0.8528	0.8172	0.5656	0.6044	0.9572	0.8204	0.8328
Over rate	0.0768	0.0028	0.0136	0.0824	0.1468	0.3580	0.3192	0.0164	0.0264	0.0184
Mean width	3.0788	3.4975	2.5061	5.5108	2.6211	3.7393	4.0583	2.5767	2.4435	21.2189
SD width	0.7257	0.8745	0.9375	15.5594	1.2302	2.7713	2.9290	1.0379	0.5760	10.1998
n=25										
Below rate	0.0700	0.0700	0.1316	0.0620	0.0276	0.0392	0.0392	0.0228	0.0836	0.1488
Cover rate	0.8592	0.8592	0.8564	0.8688	0.8424	0.6300	0.6564	0.9540	0.8592	0.8364
Over rate	0.0708	0.0708	0.0120	0.0692	0.1300	0.3308	0.3044	0.0232	0.0572	0.0148
Mean width	2.2420	2.4122	2.3007	4.1157	2.2285	2.7416	2.9575	2.1686	5.0641	15.8237
SD width	0.4268	0.4348	0.8170	9.2627	0.9808	1.5614	1.7454	0.8474	0.9639	6.1527
n=50										
Below rate	0.0596	0.0444	0.0868	0.0552	0.0240	0.0100	0.0100	0.0180	0.0716	0.1156
Cover rate	0.8768	0.8928	0.9052	0.8976	0.8720	0.6700	0.7000	0.9540	0.8592	0.8464
Over rate	0.0636	0.0628	0.0080	0.0472	0.1040	0.3200	0.2900	0.0280	0.0692	0.0380
Mean width	1.5237	1.5237	1.8825	2.1810	1.7332	1.9215	2.0238	1.6930	1.5027	11.1350
SD width	0.2070	0.2070	0.5673	1.7014	0.6389	0.8118	0.9183	0.5671	0.2041	3.0723
n=100										
Below rate	0.0512	0.0608	0.0580	0.0496	0.0156	0.0060	0.0060	0.0220	0.0000	0.1456
Cover rate	0.8880	0.9144	0.9348	0.9116	0.9084	0.7136	0.7660	0.9592	0.8812	0.8528
Over rate	0.0608	0.0248	0.0072	0.0388	0.0760	0.2804	0.2280	0.0188	0.1188	0.0016
Mean width	1.0569	1.0912	1.4382	1.4200	1.2940	1.3614	1.3921	1.2733	1.0750	7.7087
SD width	0.1029	0.0928	0.3456	0.5167	0.3783	0.4243	0.4686	0.3516	0.1047	1.5074

Notes: See Table **1**.

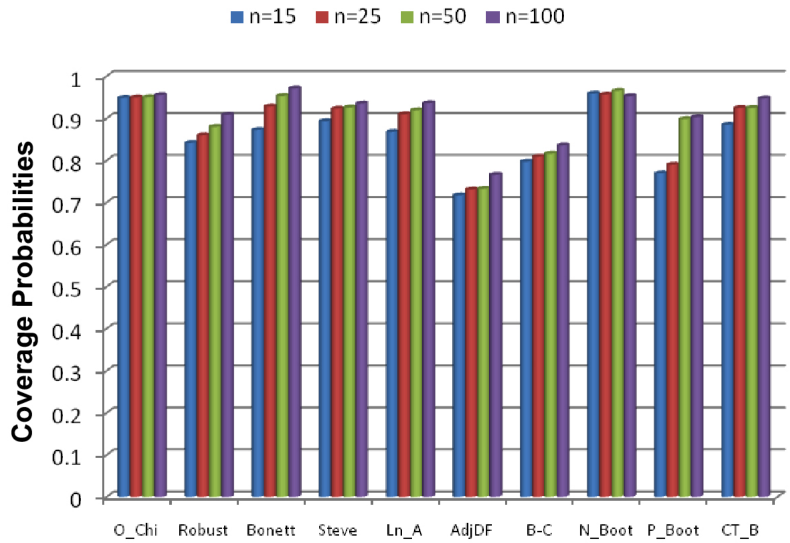

Figure 1: Coverage Probabilities for selected intervals for data from the N(0,1) distribution.

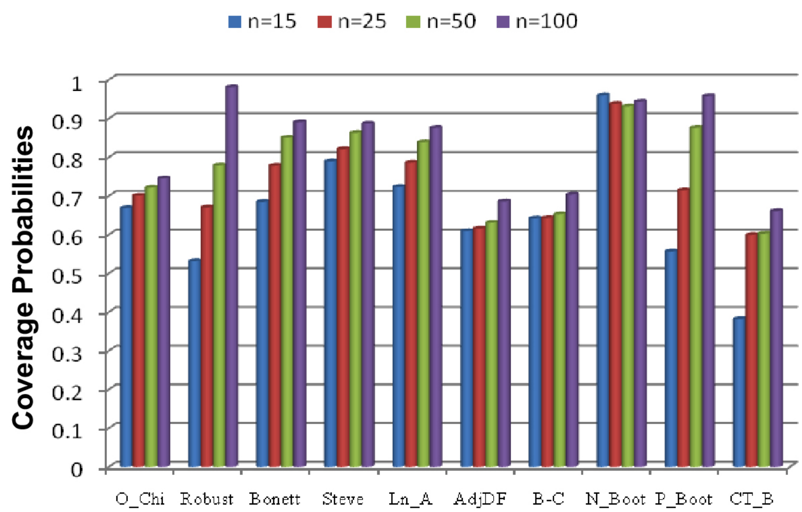

Figure 2: Coverage Probabilities for selected intervals for data from the G(3,1) distribution.

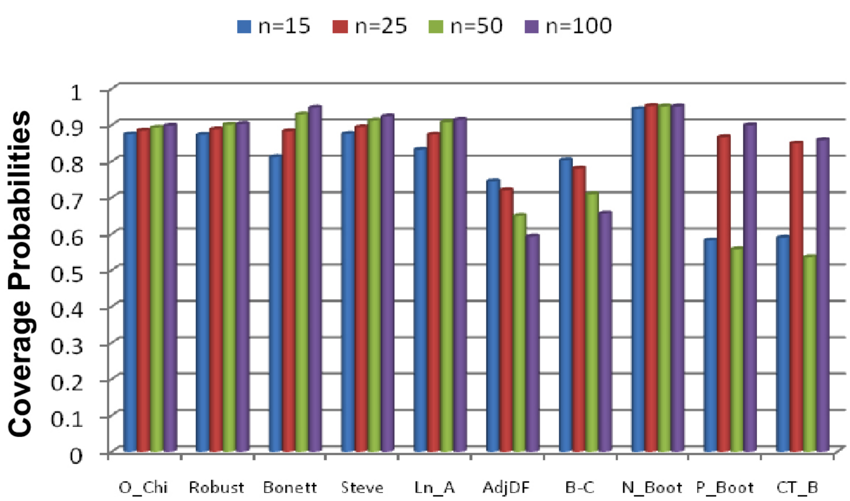

Figure 3: Coverage Probabilities for selected intervals for data from the t distribution with 8 df.

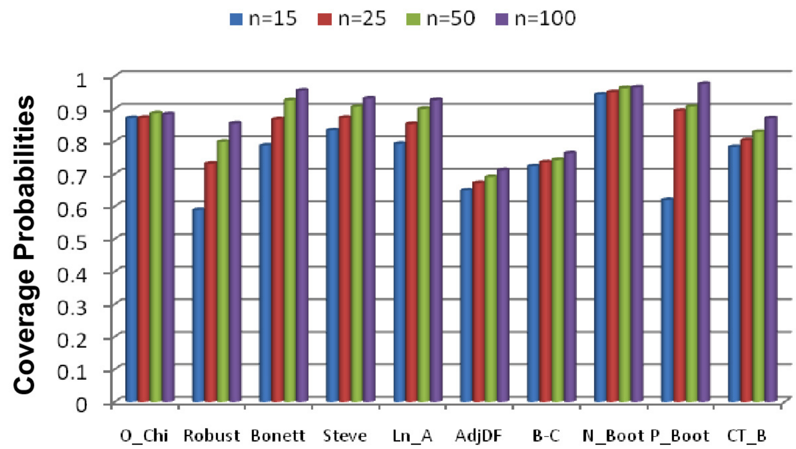

Figure 4: Coverage Probabilities for selected intervals for data from the Beta(5,1) distribution.

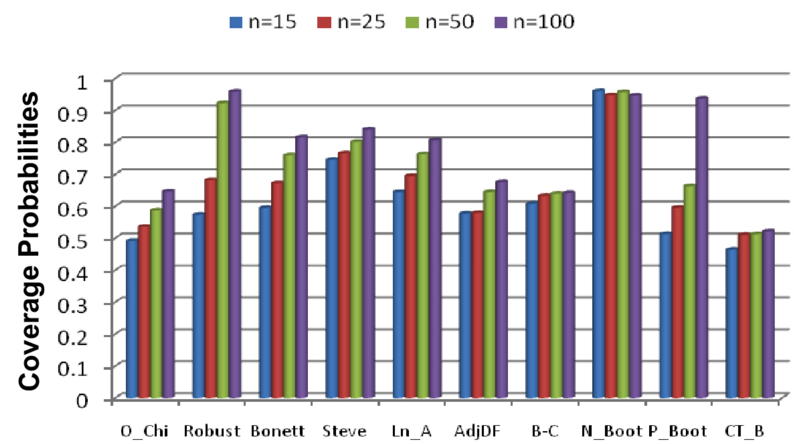

Figure 5: Coverage Probabilities for selected intervals for data from the lognormal (3,0.75) distribution.

non-parametric and the parametric intervals have coverage rate close to 0.95 for large samples. The classical chi-square, AdfDF, B-C and CT nonparametric bootstrap intervals underestimates the coverage probability.

Table **6** and Figure **6** report simulation results for the chi-square distribution with 7 df. It is observed from these results that with increasing sample sizes, coverage probability for selected intervals converge to the nominal probability of 0.95 except the AdjDF and the B-C intervals.

These two intervals although have an increase in coverage probability with an increase in the sample size n, still underestimate coverage rate.

Overall, according to our simulation results, CIs such as robust interval, Bonnet interval, Steve interval, Ln_Asymptinterval and the nonparametric bootstrap interval have the best performance and thus can be recommended for researchers.

4. REAL DATA APPLICATION

To illustrate the findings of the paper, two health related data are analyzed in this section.

4.1. Example 1

A study by Aizenberg *et al*. [15] examined the efficacy of sildenafil, a potent phospho-diesterase inhibitor in the treatment of elderly men with erectile dysfunction induced by antidepressant treatment for major depressive disorder. The ages of 10 enrollees were:

74, 81, 70, 70, 74, 77, 76, 70, 71, 72

Suppose we are interested to construct most common 95% CI for σ of ages of elderly men in the population. Sample mean, sample standard deviation and sample skewness of age are 73.5000, 3.6591 and 0.7961 respectively. The statistical software EasyFit developed by mathwave data analysis and simulations company was used to fit 60 distributions to this data

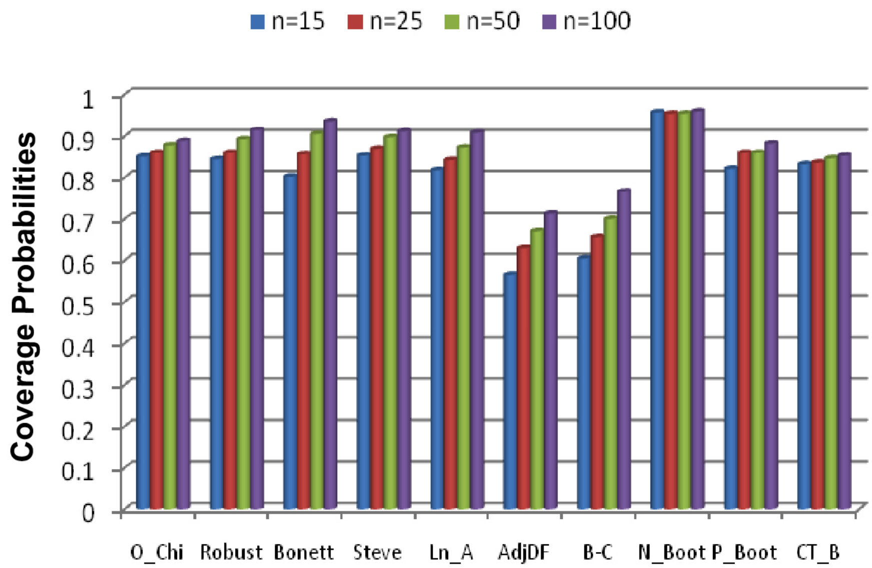

Figure 6: Coverage Probabilities for selected intervals for data from the chi-square distribution with 7 df.

set. A goodness-of-fit procedure is performed on all 60 distributions and their performances are ranked based on the goodness-of-fit measure. According to mathwave, the distribution which best fits the age data is the Gumbel Max distribution which ranked number one using the Kolmogrov-Simrnov test. The location and scale parameters of the Gumbel Max distribution obtained are μ=71.853 and σ=2.853, respectively. The variance of the Gumbel Max distribution is given by $\frac{\pi^2}{6}\sigma^2$. By substituting the scale estimate σ, an estimate of the Gumbel Max population standard deviation is $\sqrt{13.389}$=3.659. The resulting 95% CIs and corresponding confidence widths are provided in Table **7** and in the Figure **7** for better understanding. A careful inspection of those intervals reveals that all of them captured the true parameter value of 3.659 except the robust interval. Among the intervals that captured the true parameter value, Bonett interval has the narrowest width followed by the nonparametric interval and the Ln_A interval.

Table 7:Confidence Intervals Estimate of σ for Age Data Example

Method	95% CI	Width
O_Chi	(2.5168,6.6801)	4.1632
Robust	(4.4911,12.9234)	8.4323
Bonett	(2.3065,4.8535)	2.5470
Steve	(2.7303,8.1034)	5.3731
Ln_A	(2.4575,5.4481)	2.9905
AdjDF	(2.4716,7.0100)	4.5384
B-C	(2.4716,7.4909)	5.0194
N_Boot	(1.9120,4.7842)	2.8723
P_Boot	(2.8374,7.2850)	4.4476
CT_B	(1.1279,5.0016)	3.8737

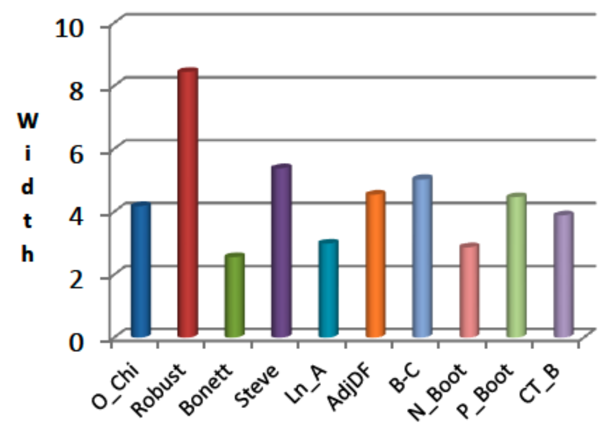

Figure 7: Confidence interval width for age data example.

4.2. Example 2

A study tested the tumorigenesis of a drug. Rats were randomly selected from litters and given the drug. The times of tumor appearance (TTA) were recorded as follows [16]:

101, 104, 104, 77, 89, 88, 104, 96, 82, 70, 89, 91, 39, 103, 93, 85, 104, 104, 81, 67, 104, 104, 104, 87, 104, 89, 78, 104, 86, 76, 103, 102, 80, 45, 94, 104, 104, 76, 80, 72, 73

Sample mean, sample standard deviation and skewness of the TTA data are 88.7805, 15.9930 and -1.21718 respectively. Similar to example 1, using EasyFit software, the distribution with the best fit was found to be the general Pareto distribution with parameters: location, scale, and shape given by μ=51.381, σ=125.8, and ξ=-2.3636, respectively. The variance of the general Pareto distribution is given by

$$Variance = \frac{\sigma^2}{(1-\xi)^2(1-2\xi)}, \xi < \frac{1}{2}$$

Thus, the variance of the general Pareto distribution is 244.2363, and hence an estimated population standard deviation of the general Pareto distribution is 15.628. We are interested inconstructing 95% CI for σ of the time until a tumor appearance. The resulting CI and corresponding confidence widths are reported in Table **8** and widths of CI are presented in Figure **8** for better understanding. A careful inspection of the intervals in Table **8** reveals that all confidence intervals captured the true parameter value of 15.628, except the robust interval. Among the nineintervals captured the true parameter value, we found that CT_B has the shortest width, followed by robust and the chi-square intervals, respectively.

Table 8: Confidence Intervals Estimates of σ for the TA Data

Method	95% CI	Width
O_Chi	(13.1304,20.4631)	7.3326
Robust	(16.1379,23.2821)	7.1442
Bonett	(11.5967,21.0496)	9.4529
Steve	(12.7921,24.1947)	11.4026
Ln_A	(12.0687,21.1932)	9.1245
AdjDF	(12.3689,22.6357)	10.2668
B-C	(12.3689,23.3562)	10.9873
N_Boot	(11.2756,20.2355)	8.9599
P_Boot	(12.6710,22.4034)	9.7324
CT_B	(12.6797,19.4429)	6.7632

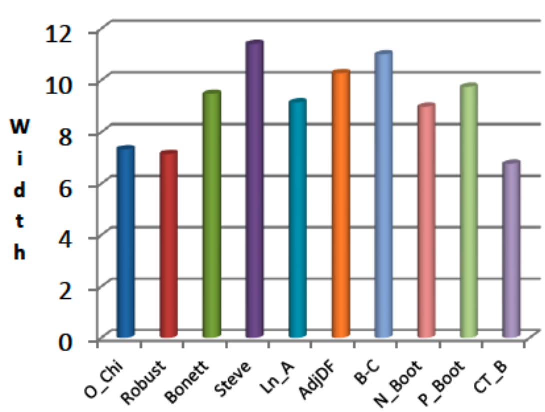

Figure 8: 95% interval width for the TA data.

5. CONCLUSION

This article generalizes the work conducted by [2]. An extensive simulation study has been conducted to compare the performance of several interval estimators for the population standard deviation. We considered the classical chi-square interval, the robust interval, the Bonett interval, the Steve interval, and three intervals proposed by [1] and three versions of bootstrap intervals. Various symmetric (light and heavy tailed) data and skewed data (positive and negatively distributed data) are evaluated. Our simulation results advise that the robust interval, the Bonnet interval, the Steve interval, the Ln_A interval, the nonparametric bootstrap intervals can be recommended for use by researchers. Some Real-life data are considered to illustrate the application of the proposed confidence intervals which also supported the simulation study to some extent.

We hope that findings of this paper will be helpful for applied researchers who want to select the proper method(s) for confidence interval for the parameter, population standard deviation.

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