Optimal Control Indicators for the Assessment of the Influence of Monetary Policy to Business Cycle Shocks

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Abstract: A linear control system in the form of a vector autoregressive (VAR) model with an input is considered. The system comprises a set of macroeconomic variables as inputs, states and outputs. The state variables included are the cyclical components of gross domestic product (GDP) and the rate of unemployment. The input variable is the three-month Central Bank official lending rate. Since all state variables are measured the state vector is also the output vector. In this setting we assess the possibility of smoothing the effect of a single large negative business cycle shock (impulse) to GDP *via* shaping the short-term nominal rate of interest and propose optimal control indicators measuring the control potential of this Central Bank action. The results obtained indicate that the actions of the Central Bank improved the performance of the dynamical system, a fact reflected on the indicators. These results contribute to the existing knowledge on the effectiveness of monetary policy as a short-run stabilization device which is still an open issue.

Keywords: Control system, business cycle shock, optimal control, optimal control indicators.

1. INTRODUCTION

One of the main tasks for macroeconomists is to explain how macroeconomic aggregates, such as GDP, unemployment and the price level, behave over time. The movement of these aggregates is determined by economic policy and the changes in the economic environment. Monetary and fiscal policies seek to minimize the unwanted fluctuations of macroeconomic aggregates through the use of various policy instruments such as nominal interest rates, government spending and taxes. In order to examine the effectiveness of such policies one has to employ empirical methods utilizing observable data for the various macroeconomic variables.

Prior to Sims (1980) the classical, traditional empirical method in macroeconomics was to statistically estimate large-scale linear macroeconomic models typically based around the Keynesian macroeconomic model. However this practice required the imposition of a large number of identifying restrictions on the structural parameters of the model to such an extent that the resulting interpretations, forecasts and policy conclusions were almost inevitably flawed and thus of little use to the policy maker.

As an alternative, Sims proposed that the empirical study of macroeconomic variables could be based on the use of vector autoregressive (VAR) models. A VAR model is a stochastic linear time-invariant vector difference equation and can be regarded as the reduced form of an econometric system of structural equations in which there are no exogenous variables but only lagged endogenous variables. The most common use of VAR models is in short-run forecasts for which the knowledge of the structural parameters is not required. Only the reduced form parameters are required whose estimation is always feasible using ordinary least squares (OLS). The formulation of a VAR model lacks theoretical basis, i.e it is *atheoretical*, in the sense that there is no explicit formulation of structural equations that arise from economic theory.

From the point of view of control theory, a VAR model is termed as uncontrolled since there is no input that can be selected in order to control the output of the system. In each case a shock occurs to some variable(s) and from that point onwards the response of the system is determined by its uncontrolled dynamics. VAR models have been used extensively in order to understand the empirical effects of monetary policy on real economic activity. Sims (1972, 1980) pioneered the use of VAR models to estimate the impact of money on the economy. Christiano, Eichenbaum and Evans (1998) provided a thorough discussion of the use of VAR models to estimate the impact of money, and they provided an extensive list of references to work in this area. More recently VAR models have also been used in order to study the effects of fiscal policy (Blanchard and Perotti, 2002, Giordano, Momigliano, Neri, and Perotti, 2007).

In this paper we begin by formulating a control system in the form of a VAR model. The state variables considered for that matter are the cyclical components of GDP and the rate of unemployment. The control variable is the three-month Central Bank official lending rate. A nominal interest rate along with a measure of

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the money stock are typical measures of monetary policy (Woodford, 2003). Since all state variables are measured the state vector is also the output vector. After least squares identification of the controlled vector autoregressive model we solve the optimal control problem of minimizing the effect of a single large negative business cycle shock (impulse) to GDP *via* shaping the nominal interest rate. The cost criterion considered is $\sum_n y_n^2$, where y_n denotes the deviation of log GDP from its trend at time point n. The decomposition of GDP and the rate of unemployment into a trend component and a cyclical component can be realized by using a filter such as the Hodrick-Prescott (HP) filter.

The solution of the problem is based on the following two assumptions: (a) A large negative shock to GDP equal to the minimum value of the residuals distribution of that particular state variable. (b) A nonnegativity constraint on the control variable, i.e. the nominal interest rate, at each time point n. An economy is said to be in a "liquidity trap" when the monetary authority, i.e. the Central Bank, cannot achieve a lower nominal interest rate in order to stimulate output. This situation can arise when the nominal interest rate has reached its zero lower bound, below which nobody would be willing to lend if money can be stored at no cost for a nominally risk-free zero rate of return. In the economics literature, the nonnegativity constraint on the nominal interest rate is usually refered to as the zero lower bound (ZLB) or zero floor assumption and has been studied extensively by various authors (Iwata and Wu, 2006, Blinder, 2000, Bernanke and Reinhart, 2004). The control schemes considered are both open-loop and closed-loop. In the first case, an open-loop optimal control is to be determined that minimizes the cost criterion subject to the controlled vector autoregressive model and the control nonnegativity constraint. In the open-loop approach the Central Bank sets the interest rate based on a specified initial state of the economy. In the second case a closed-loop optimal control is to be determined in the form of a dynamic linear time invariant state feedback controller. In this case the Central Bank determines the optimal interest rate based on *any* state of the economy and not just a particular initial state. The solution to both these problems will provide us with a variety of optimal control indicators that will help us assess the control potential of the Central Bank's actions versus a single large negative business cycle shock to GDP.

The effectiveness of monetary policy as a short-run stabilization device has been studied by a large number of authors over the past decades. Fair (2005) uses optimal control techniques in order to address the issue. The control system he considers is the United States subset of the MC model (Fair, 1994) with minor modifications. The performance index to be minimized reflects the goal of the Federal Reserve which is assumed to care about deviations of GDP and inflation from certain base values as well as interest rate fluctuations. The input of the system is the three-month Treasury bill rate. The optimization problem at hand consists of choosing optimal values of the input that minimize the performance index subject to the model used and the presence of random shocks. These are drawn from the estimated error terms in the equations of the econometric model and are taken as estimates of the economic shocks. Rasche and Williams (2005) provide a summary of the findings on monetary policy effectiveness for the United States. Angeloni and Ehrmann (2003) examine the monetary policy transmission mechanism in euro area countries and also provide a selective review of the literature. Cecchetti, Flores-Lagunes and Krause (2006) study a sample of 24 countries, ranging from large industrial countries to small developing ones. They find that in 21 of the 24 countries studied more efficient monetary policy has been the driving force behind improved macroeconomic performance. The most popular empirical method used for addressing the issue of monetary policy effectiveness has been vector autoregressions.

In spite of the fact that VAR models have been a popular tool for studying the monetary transmission mechanism, i.e. the process through which monetary policy decisions are transmitted into changes in real GDP and inflation, the coefficients of these models and their statistical properties do not reveal the ability of one variable (input) to control the other (output). This information is hidden within the coefficients of the model and can only be recovered through the study of the dynamic relationship as an input-output control system. To this end, after least squares identification of the controlled vector autoregressive model we solve the optimal control problem of minimizing the effect of a single large negative business cycle shock to GDP subject to the constraints imposed by the controlled vector autoregressive model and the zero lower bound condition on the control variable. An open-loop optimal control (OLOC) as well as a closed-loop optimal control (CLOC) are determined for this matter.

2. PROBLEM DEFINITION

The economic problem at hand is to develop a method for examining the effectiveness of monetary policy as a short-run output stabilization device. This is a long-standing issue in the literature of monetary economics and Central Banking and the existing evidence may still be termed as inconclusive (Poole, 2004). The significance of the problem at hand is evident given that output stabilization is a key goal of economic policy. The approach taken here is based on the use of optimal control. The aim is to determine and, subsequently, calculate a set of optimal control indicators derived from the solution of both an open-loop and a closed-loop optimal control problem. These indicators will be used in order to provide some answers to the economic question. We will proceed in this section by outlining the general framework of the problem to be examined and give the specifics in the next section (section 3) on problem methodology. Consider the following controlled vector autoregressive model in state space form modeling the dynamics of a macroeconomic system:

$$x_n = c + \Phi_1 x_{n-1} + \Phi_2 x_{n-2} + ... + \Phi_p x_{n-p} + Bu_n + \varepsilon_n = \qquad (1)$$

The selection of this form of model is based on the simple fact that vector autoregressions constitute the econometric model of choice when it comes to understanding the empirical effects of monetary policy on the economy. Using lag operator notation, (1) can be written in the form

$$(I_k - \Phi_1 L - \Phi_2 L^2 - ... - \Phi_p L^p) x_n = c + Bu_n + \varepsilon_n$$

or

$$\Phi(L) x_n = c + Bu_n + \varepsilon_n ,$$

where $\Phi(L)$ indicates the $k \times k$ matrix polynomial in the lag operator L defined by $I_k - \Phi_1 L - \Phi_2 L^2 - ... - \Phi_p L^p$. The matrices Φ_1, Φ_2,..., Φ_p are $k \times k$ system or dynamic matrices. Matrix B is the $k \times m$ policy matrix, x_n is the vector of macroeconomic variables, u_n is the vector of policy variables, c is a constant vector and finally ε_n is a vector white noise satisfying the properties:

$$E(\varepsilon_n) = 0 , \quad E(\varepsilon_n \varepsilon_s^t) = \begin{cases} V, s = n \\ 0, s \neq n \end{cases} ,$$

where V is a $k \times k$ symmetric, positive definite variance-covariance matrix and t denotes transposition (Hamilton, 1994). System (1) is asymptotically stable if and only if all the roots of the determinantal equation $| I_k \lambda^p - \Phi_1 \lambda^{p-1} - \Phi_2 \lambda^{p-2} - ... - \Phi_p | = 0$ have modulus strictly less than unity. We may also have algebraic constraints of the type $g(x_n, u_n) \succeq b$, where g is some vector-valued function and $\succeq$ denotes componentwise inequality between vectors. The performance of the uncompensated open-loop system (1) is to be modulated to minimize some performance index (PI) which we assume to be a 2-norm penalizing the deviations of a target variable from its desired path or trajectory. Given our stated aim of evaluating monetary policy as a short-run output stabilization device it's natural to consider the cyclical component of GDP as the economic variable to be controlled. The solution is to be such that the PI takes on its minimum for some $u_n \in \Omega \subset \Re^m$, where Ω is the set of all admissible policies, i.e. policies that satisfy the constraints of the problem, subject to the controlled vector autoregressive model describing the dynamics of the macroeconomic system and the specified inequality constraints. An optimal policy as a function of the states ,i.e. in closed-loop form, is also to be designed.

The definition of the problem depends on the assumptions made about the shocks. If ε_n is a single large negative shock (impulse) then u_n must be designed in such way as to minimize the effects of this shock. If ε_n are periodic negative shocks then the optimal control law has to be determined accordingly. Here we consider ε_n to be an impulse. Under this assumption the problem makes sense for a short-run period spanning 15-20 quarters. The solution of the problem, comprising both an open-loop and closed-loop optimal control, will provide us with a variety of indicators that will allow us to evaluate the control potential of the Central Bank's actions versus a large negative business cycle shock to GDP. These indicators are: (i) The minimum value of the PI. This value depends on the parameters and the form of the system, the size and type of the shocks, the form and nature of the inequality constraints and the form and nature of the PI. It is obvious that this value incorporates a lot of information about the system's potential controllability. (ii) The magnitude of the largest open-loop and closed-loop system poles. These provide information on the speed of absorption of the impulse shock. The smaller the magnitude, and certainly less than unity to make sense, the faster the absorption of the shock by the system. (iii) Percentage reduction of the largest uncompensated open-loop system pole magnitude when state feedback is applied.

(iv) Percentage reduction of the optimal value of the PI when feedback is applied as opposed to the uncontrolled system ($u_n \equiv 0$).

3. PROBLEM METHODOLOGY

The PI considered is $\sum_{n=0}^{T-1} y_n^2$, where y_n denotes the deviation of log GDP from its trend and T is the problem's time horizon. The choice of the PI reflects the short-run goal of monetary policy, as practiced by Central Banks, which is to counteract GDP fluctuations compared to price level stability in the long-run. The quadratic term y_n^2 inside the sum penalizes the deviations. The additive decomposition of log GDP and the unemployment rate into a trend component and a cyclical component is carried out using the Hodrick-Prescott (HP) filter. The state equation considered here is

$$x_n = \Phi_1 x_{n-1} + Bu_n + \varepsilon_n \quad = \qquad (2)$$

and is identified using least squares identification. The choice of lag length was made based on minimizing the Schwarz (SC) information criterion after running the model for lag lengths starting from one up to four given that the data are quarterly. A residual serial correlation Lagrange multiplier (LM) test was performed for each of the models estimated which led to the acceptance of the null in all three cases. The equation for output w_n is simply $w_n = x_n$ since all state variables are measured. The component-sequences of the state vector x_n are the cyclical components of GDP (y_n) and the unemployment rate (U_n). Hence $x_n = \left(y_n U_n\right)^t$. The policy variable is the three-month Central Bank official lending rate denoted by i_n. The ZLB or zero floor constraint on the policy variable is $i_n \geq 0$. The OLOC is determined as a function of time n for a specified initial state value. Hence the dynamic optimization problem becomes:

$$\underset{\{i_n\}_{n=1}^{T-1}}{Minimize} \sum_{n=0}^{T-1} y_n^2$$

$$s.t. \ \begin{pmatrix} y_n \\ U_n \end{pmatrix} = \Phi_1 \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + Bi_n + \begin{pmatrix} \varepsilon_n^{(1)} \\ \varepsilon_n^{(2)} \end{pmatrix}$$

$$i_n \geq 0$$

$$y_{-1} = 0$$

$$U_{-1} = 0$$

Φ_1 is the 2×2 system or dynamic matrix and B is the 2×1 policy matrix. The target set is $S = \{T-1\} \times \Re^2$. The following assumptions are made:

(a) = Before the appearance of the shock the economy follows the trend i.e. $y_{-1} = U_{-1} = 0$.

(b) = A single large negative business cycle shock occurs to GDP at time n i.e. $\varepsilon_n = \begin{cases} \varepsilon, & n = 0 \\ 0, & n \neq 0 \end{cases}$, where ε is the constant vector $\varepsilon = \begin{pmatrix} \varepsilon_0^{(1)} \\ 0 \end{pmatrix}$. This assumption is key since we want to evaluate the effectiveness of monetary policy as a short-run output stabilization device. Therefore it is only appropriate to consider a strong negative shock to GDP and examine how monetary policy can fare given such a large shock.

(c) = The Central Bank acts one period after the appearance of the shock. This assumption is plausible since monetary policy, or even fiscal policy for that matter, works with a delay i.e. a shock has to develop before the policy maker acts.

Under these assumptions the preceding open-loop optimal control problem may be rewritten as:

$$\underset{u}{Minimize} \ \varepsilon^t D\varepsilon + x^t Px$$

$$s.t. \quad Au = Cx + h$$

$$u \succeq 0$$

(the details are provided in Appendix A). The above problem may be solved by convex quadratic programming (QP) methods (Boyd, Vandenberghe, 2004). The CLOC is determined as a function of the cyclical components of GDP and the unemployment rate lagged one period. Hence we consider the following dynamic linear time invariant state feedback controller:

$$i_n = Kx_{n-1}$$

Note that dynamic controllers arise naturally in economics since economic policy, be it fiscal or monetary, is characterized by the presence of lags. An advanced reference on the use of feedback policy in an economic system is Kotsios and Leventides (2004). The compensated closed-loop system then is:

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = (\Phi_1 + BK)\begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_n^{(1)} \\ \varepsilon_n^{(2)} \end{pmatrix}$$

The economic interpretation of such a system is the following: The policy maker, in this case the Central Bank, feeds back on information about the lagged state of the economy and adjusts the decisions taken based on that information. The controller accepts the deviations of log GDP and the unemployment rate from their respective trends and decides the proper corrective action to minimize the PI. The output of the controller, i.e. the short-term Central Bank official lending rate, is then applied to the process or final control element i.e. the macroeconomy in this case. Hence the dynamic optimization problem is put forward in the following manner:

$$\underset{K}{Minimize} \sum_{n=0}^{T-1} y_n^2$$

$$s.t. \begin{pmatrix} y_n \\ U_n \end{pmatrix} = (\Phi_1 + BK)\begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_n^{(1)} \\ \varepsilon_n^{(2)} \end{pmatrix}$$

$$K\begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} \geq 0 \text{ ,}$$

where $K = \left(k_1\, k_2\right)$ is a 1×2 feedback matrix of constant gains (to be designed). Following some manipulations this problem may be rewritten as:

$$\underset{k_1,k_2}{Minimize} \sum_{i=0}^{T-1} w_i (k_1, k_2)^2$$

$$s.t.\, K\varepsilon \geq 0$$

$$Kx_i \geq 0$$

(see Appendix A). The solutions to both the open-loop and closed-loop design schemes will provide information on the potential for monetary policy to control the effects of a large negative business shock to GDP. This information is captured by the optimal control indicators proposed and can be used to refine the Central Bank's analysis concerning the impact of its actions on cyclical GDP following a strong negative business cycle shock.

4. QUANTITATIVE RESULTS

The following results in Table **1** were obtained using a sample of thirty-five quarterly observations for the period 1999:1 - 2007:3 (eurostat database) for a set of three countries, namely, the Czech Republic, Japan and the United Kingdom. The selection of these countries was based on the fact that they are not members of any common monetary union. If they were, then there would be no point in making any comparisons between them with respect to monetary policy since the latter would be common. Both GDP and the rate of unemployment were HP-detrended following deseasonalization. The poles of the uncompensated open-loop system as well as those of the compensated closed-loop system lie inside the unit circle in all cases i.e. these are asymptotically stable linear dynamical systems. The application of dynamic state feedback improved the response of the uncompensated open-loop system by reducing the magnitude of its largest pole. The reduction is largest, equal to -20.51%, in the case of the uncompensated system identified for the United Kingdom. The error, i.e. the optimal value of the PI, is also smallest, equal to 0.002968, under closed-loop control in the case of the UK. Note that Japan has the largest error compared to the Czech Republic and the United Kingdom and this is

Table 1: = Quantitative Results

Country	Czech Republic	Japan	United Kingdom
Largest open - loop system pole magnitude	0.815	0.787	0.790
Largest closed - loop system pole magnitude	0.797	0.743	0.628
Error: Open - loop system	0.00137	0.005619	0.002116
Error: Closed - loop system	0.003159	0.00788	0.002968
Error: Uncontrolled system	0.003874	0.011516	0.003464
Error reduction (%)	-18.46%	-31.57%	-14.32%
Largest open-loop system pole magnitude reduction (%)	-2.21%	-5.59%	-20.51%

true for both the open-loop and closed-loop control cases. This means that the control ability of the system in the case of Japan is limited compared to the Czech Republic and the United Kingdom. This may not be surprising given that for the past two decades or so Japan has been in a liquidity trap which means that the ability of monetary policy to counteract a large negative business cycle shock to GDP is constrained. Our results confirm this fact.

Case I-Czech Republic

Least squares identification yielded the following results:

Table 2: = Least Squares Identification Results: Case I-Czech Republic

Vector Autoregression Estimates		
Sample (adjusted): 1999Q2 2007Q3		
Included observations: 34 after adjustments		
Standard errors in ( ) & t-statistics in [ ]		
	GDPCZESACY	UNEMCZESACY
GDPCZESACY(-1)	0.818264	-3.634232
	(0.13933)	(1.86819)
	[5.87287]	[-1.94532]
UNEMCZESACY(-1)	0.001836	0.803756
	(0.00785)	(0.10532)
	[0.23376]	[7.63174]
IRCZE	0.000205	-4.31E-06
	(0.00061)	(0.00819)
	[0.33607]	[-0.00053]
R-squared	0.636847	0.811441
Adj. R-squared	0.613418	0.799276
Sum sq. resids	0.010134	1.821933
S.E. equation	0.018080	0.242429
F-statistic	27.18171	66.70224
Log likelihood	89.76601	1.505952
Akaike AIC	-5.103883	0.087885
Schwarz SC	-4.969204	0.222564
Mean dependent	0.000156	0.018221
S.D. dependent	0.029079	0.541110
Determinant resid covariance (dof adj.)		1.65E-05
Determinant resid covariance		1.37E-05
Log likelihood		93.84882
Akaike information criterion		-5.167578
Schwarz criterion		-4.898220

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = \begin{pmatrix} 0.818264 & 0.001836 \\ -3.634232 & 0.803756 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} 0.000205 \\ -0.00000431 \end{pmatrix} i_n + \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix}$$

The eigenvalues of the system matrix $\begin{pmatrix} 0.818264 & 0.001836 \\ -3.634232 & 0.803756 \end{pmatrix}$ are $0.811 \pm 0.081i$ with magnitude 0.815 thus lying inside the unit circle. Equivalently the state space pair (Φ_1, B) is asymptotically stable. The full table of least squares identification results is provided in Table **2**.

The size of the impulse shock to GDP considered is equal to -0.037. This is equal to the minimum value of the residuals distribution from the first equation of the system for the cyclical component of GDP. The graph of this distribution is provided in Figure **1**.

The dynamic state feedback controller constructed is:

$$i_n = -218.724 y_{n-1} + 7.951419 U_{n-1}$$

Hence the compensated closed-loop system is the following:

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = \begin{pmatrix} 0.773426 & 0.003466 \\ -3.633289 & 0.803722 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix}$$

The eigenvalues of the closed-loop system matrix $\begin{pmatrix} 0.773426 & 0.003466 \\ -3.633289 & 0.803722 \end{pmatrix}$ are $0.789 \pm 0.111i$ having modulus 0.797 thus lying inside the unit circle. Equivalently the compensated closed-loop system is asymptotically stable. The graph of the impulse response for the uncompensated open-loop system and the compensated closed-loop system, after the application of optimal policy, as well as the uncontrolled system is given in Figure **2**.

The optimal value of the PI, i.e. the error, in the case of open-loop control is equal to 0.00137. In the case of closed-loop control the optimal (minimum) value of the PI is equal to 0.003159. Finally in the uncontrolled case the value the PI equals 0.003874.

Case II-Japan

Least squares (parametric) identification of the uncompensated open-loop system yielded the following results:

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = \begin{pmatrix} 0.720066 & -0.038335 \\ -0.067450 & 0.747844 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} 0.020874 \\ -0.044647 \end{pmatrix} i_n + \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix}$$

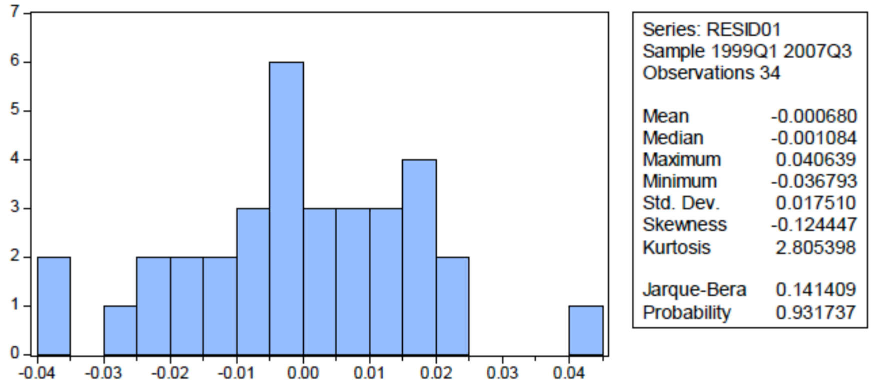

Figure 1: Residuals distribution from the estimation of the first equation in the case of the Czech Republic.

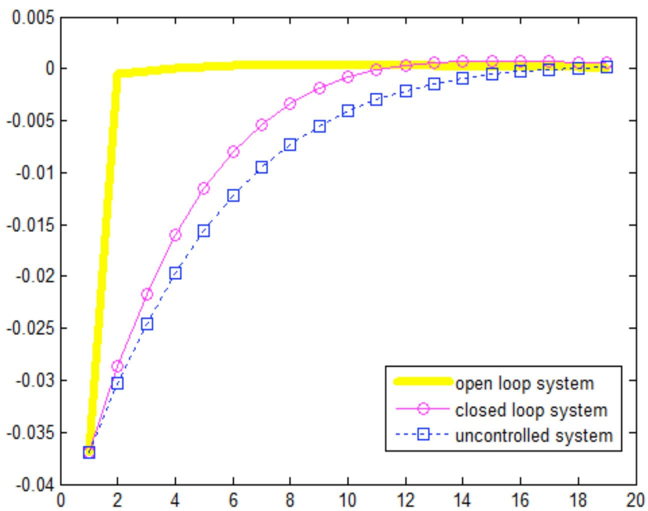

Figure 2: Impulse response: Case I-Czech Republic.

The system or dynamic matrix $\begin{pmatrix} 0.720066 & -0.038335 \\ -0.067450 & 0.747844 \end{pmatrix}$ is Schur stable with eigenvalues 0.787 and 0.681 lying inside the unit circle. The full table of least squares parametric identification results is provided in Table **3**.

The size of the shock to GDP, considered for the purpose of designing the OLOC and the CLOC, is equal to -0.074. This is the minimum value of the residuals distribution from the first equation of the controlled VAR model for the state variable y_n. The graph of the distribution is provided in Figure **3**.

The CLOC constructed is

$$i_n = -8.7515 y_{n-1} - 4.47821 U_{n-1}$$

Hence the compensated closed-loop system, given the gain state matrix $K = (-8.7515 - 4.47821)$ designed, is the following:

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = \begin{pmatrix} 0.537387 & -0.131813 \\ 0.323278 & 0.947783 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix}$$

The closed-loop system matrix $\begin{pmatrix} 0.537387 & -0.131813 \\ 0.323278 & 0.947783 \end{pmatrix}$ is Schur stable having

Table 3: = Least Squares Identification Results: Case II-Japan

Vector Autoregression Estimates		
Sample (adjusted): 1999Q2 2007Q3		
Included observations: 34 after adjustments		
Standard errors in ( ) & t-statistics in [ ]		
	GDPJAPSACY	UNEMJAPSACY
GDPJAPSACY(-1)	0.720066	-0.067450
	(0.08849)	(0.33093)
	[8.13704]	[-0.20382]
UNEMJAPSACY(-1)	-0.038335	0.747844
	(0.03292)	(0.12312)
	[-1.16441]	[6.07413]
IRJAP	0.020874	-0.044647
	(0.01699)	(0.06352)
	[1.22883]	[-0.70283]
R-squared	0.738916	0.626260
Adj. R-squared	0.722072	0.602147
Sum sq. resids	0.032338	0.452249
S.E. equation	0.032298	0.120784
F-statistic	43.86789	25.97263
Log likelihood	70.03995	25.19410
Akaike AIC	-3.943526	-1.305535
Schwarz SC	-3.808848	-1.170856
Mean dependent	0.005103	0.005353
S.D. dependent	0.061265	0.191490
Determinant resid covariance (dof adj.)		1.36E-05
Determinant resid covariance		1.13E-05
Log likelihood		97.19800
Akaike information criterion		-5.364588
Schwarz criterion		-5.095230

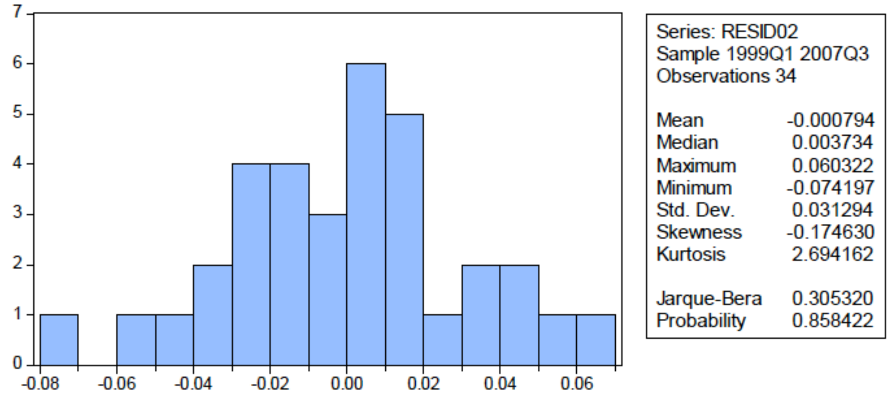

Figure 3: Residuals distribution from the estimation of the first equation in the case of Japan.

eigenvalues the complex conjugate pair $0.742 \pm 0.022i$ with magnitude 0.743. The graph of the impulse response, after the application of optimal policy, for the cases of open-loop control and closed-loop control as well as the uncontrolled case is provided below:

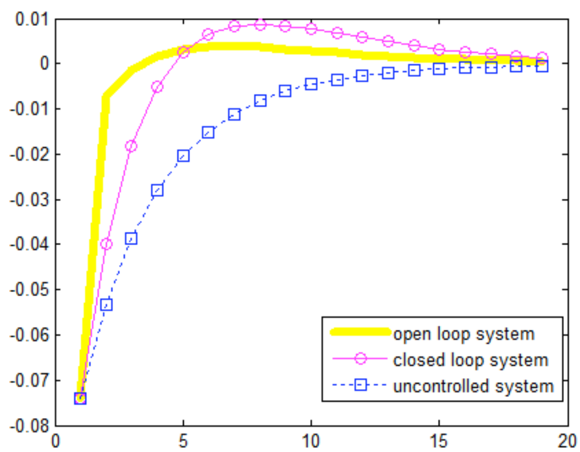

Figure 4: Impulse response: Case II-Japan.

The optimal value of the PI in the case of open-loop control is 0.005619. In the case of closed-loop control the optimal value of the cost criterion is equal to 0.00788. Finally, in the case of the uncontrolled system the value of the PI is equal to 0.011516.

Case III-United Kingdom

Parametric identification, using least squares, of the uncompensated open-loop system yielded the following results:

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = \begin{pmatrix} 0.624483 & 0.004464 \\ -0.114694 & 0.793404 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} 0.000898 \\ -0.003140 \end{pmatrix} i_n + \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix}$$

The system or dynamic matrix $\begin{pmatrix} 0.624483 & 0.004464 \\ -0.114694 & 0.793404 \end{pmatrix}$ is Schur stable having eigenvalues 0.790 and 0.628 lying inside the unit circle.

The full least squares parametric identification results for the state space pair (Φ_1, B) are provided in Table **4**.

Table 4: = Least Squares Identification Results: Case III-United Kingdom

Vector Autoregression Estimates		
Sample (adjusted): 1999Q2 2007Q3		
Included observations: 34 after adjustments		
Standard errors in ( ) & t-statistics in [ ]		
	GDPUNISACY	UNEMUNISACY
GDPUNISACY(-1)	0.624483	-0.114694
	(0.10602)	(0.63556)
	[5.89046]	[-0.18046]
UNEMUNISACY(-1)	0.004464	0.793404
	(0.01622)	(0.09721)
	[0.27526]	[8.16147]
IRUNI	0.000898	-0.003140
	(0.00070)	(0.00418)
	[1.28863]	[-0.75134]
R-squared	0.541506	0.689156
Adj. R-squared	0.511925	0.669101
Sum sq. resids	0.011890	0.427325
S.E. equation	0.019585	0.117408
F-statistic	18.30630	34.36422
Log likelihood	87.04877	26.15778
Akaike AIC	-4.944045	-1.362223
Schwarz SC	-4.809367	-1.227544
Mean dependent	0.002792	-0.009927
S.D. dependent	0.028033	0.204104
Determinant resid covariance (dof adj.)		5.07E-06
Determinant resid covariance		4.22E-06
Log likelihood		113.9064
Akaike information criterion		-6.347433
Schwarz criterion		-6.078075

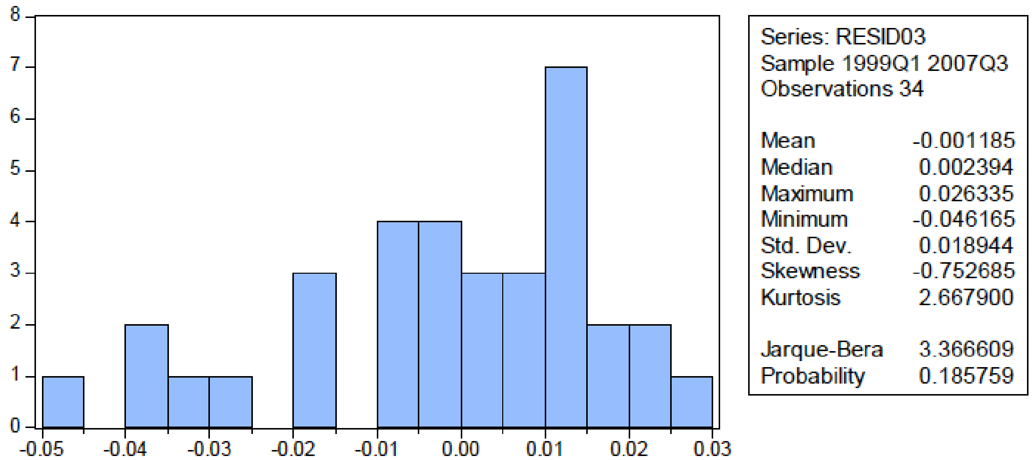

Figure 5: Residuals distribution from the estimation of the first equation in the case of the United Kingdom.

The size of the shock to GDP, considered for the purpose of determining the OLOC and CLOC, equals -0.046 which is the minimum value of the residuals distribution from the first equation of the controlled VAR model for the state variable y_n. The graph of the distribution is given in Figure **5**.

The CLOC constructed is

$$i_n = -145.331 y_{n-1} + 210.1305 U_{n-1}$$

Hence the compensated closed-loop system is the following:

$$\begin{pmatrix} y_n \\ U_n \end{pmatrix} = \begin{pmatrix} 0.493976 & 0.193161 \\ 0.341645 & 0.133594 \end{pmatrix} \begin{pmatrix} y_{n-1} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} e_n^{(1)} \\ e_n^{(2)} \end{pmatrix}$$

The closed-loop system matrix $\begin{pmatrix} 0.493976 & 0.193161 \\ 0.341645 & 0.133594 \end{pmatrix}$ is Schur stable with eigenvalues 0.628 and 0.00000041. The graph of the impulse response for the uncompensated open-loop system and the compensated closed-loop system, after the application of optimal policy, as well as the uncontrolled system is given in Figure **6**.

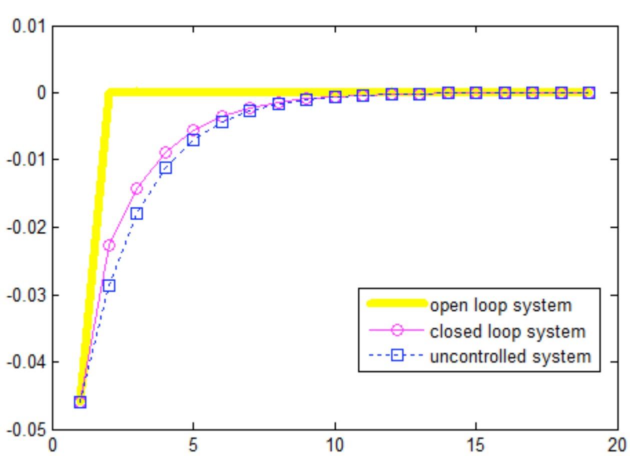

Figure 6: Impulse response: Case III-United Kingdom.

The optimal value of the PI, i.e. the error, in the case of open-loop control is equal to 0.002116. In the case of closed-loop control the optimal (minimum) value of the PI is equal to 0.002968. Finally in the uncontrolled case the value the PI equals 0.003464.

5. CONCLUSIONS

Optimal control methods were applied in order to assess the control potential of Central Bank action in smoothing the effect of a single large negative business cycle shock to GDP *via* shaping a short-term nominal rate of interest. A set of optimal control indicators were calculated for that matter. The actions of the Central Bank apparently improved the performance of the dynamical system, a fact reflected on the indicators. These indicators may be used as a measure of the control ability of monetary policy, as practised by Central Banks, to counteract adverse short-run GDP fluctuations and refine the information obtained by VAR analysis. The proposed indicators are: (i) The minimum value of the PI in both the open-loop and closed-loop approaches to designing the input. (ii) The magnitude of the largest open-loop and closed-loop system poles. (iii) Percentage reduction of the largest uncompensated open-loop system pole magnitude when feedback is applied. (iv) Percentage reduction of the optimal value of the PI when feedback is applied compared to the uncontrolled system. Finally, these indicators were calculated for three countries.

APPENDIX A.

Under the assumptions (a), (b) and (c) outlined in section 3 on problem methodology the constraints $x_n = \Phi_1 x_{n-1} + Bu_n + \varepsilon_n$ and $u_n \geq 0$ of the open-loop optimal control problem can be rewritten as:

$$\begin{array}{ll} x_0 = \varepsilon & u_1 \geq 0 \\ x_1 = \Phi_1 x_0 + Bu_1 & u_2 \geq 0 \\ x_2 = \Phi_1 x_1 + Bu_2 & u_3 \geq 0 \\ \vdots & \vdots \\ x_{T-1} = Ax_{T-2} + Bu_{T-1} & u_{T-1} \geq 0 \end{array}$$

Define the following:

$$A := \begin{pmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B \end{pmatrix}, \text{ C:=} \begin{pmatrix} I_2 & 0 & \dots & 0 \\ -\Phi_1 & I_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_2 \end{pmatrix}, \text{ P:=} \begin{pmatrix} D & 0 & \dots & 0 \\ 0 & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{pmatrix}$$

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{T-1} \end{pmatrix}, \ u := \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{T-1} \end{pmatrix}, \text{ h:=} \begin{pmatrix} -\Phi_1 \varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ D:=} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Keeping in mind that $\varepsilon = \begin{pmatrix} \varepsilon_0^{(1)} \\ 0 \end{pmatrix}$ the initial open-loop optimal control problem can be rewritten as:

$$\underset{u}{Minimize} \ \ \varepsilon^t D \varepsilon + x^t P x$$

$$s.t. \quad Au = Cx + h$$

$$u \succeq 0$$

which is a problem in convex QP. As for the closed-loop optimal control problem, the compensated closed-loop system, written compactly as,

$$x_n = (\Phi_1 + BK)x_{n-1} + \varepsilon_n$$

yields the following set of linear equations:

$$T(k_1, k_2)x = g \ ,$$

where

$$T(k_1,k_2) = \begin{pmatrix} I_2 & 0 & 0 & \dots & 0 \\ -(\Phi_1 + BK) & I_2 & 0 & \dots & 0 \\ 0 & -(\Phi_1 + BK) & I_2 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -(\Phi_1 + BK) & I_2 \end{pmatrix}, \ g = \begin{pmatrix} (\Phi_1 + BK)\varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and x as previously defined. This is subject to the constraints:

$K\varepsilon \geq 0$ and $Kx_i \geq 0,\ i=1,2,...,T-2$

The linear equations may easily be solved as

$$x = T(k_1,k_2)^{-1} g$$

with

$$x_i = \left(w_i(k_1,k_2),\ \mathrm{v}_i(k_1,k_2)\right)^t,\ i{=}1,2,...,T-1$$

And the optimization problem at hand becomes:

$$\underset{k_1,k_2}{Minimize} \sum_{i=0}^{T-1} w_i(k_1,k_2)^2$$

$$s.t.\ K\varepsilon \geq 0$$

$$Kx_i \geq 0$$

as in section 3.

APPENDIX B. LIST OF SYMBOLS

Symbol	=	Description
y_n	=	Deviation of log GDP from its trend at time n
x_n	=	Vector of macroeconomic variables
u_n	=	Vector of policy variables
c, ε	=	Constant vectors
ε_n	=	Vector white noise
U_n	**=**	Cyclical component of unemployment rate at time n
i_n	=	Three-month Central Bank official lending rate
$\Phi_1, \Phi_2, ..., \Phi_p$	=	System or dynamic matrices
L	=	Lag operator
K	=	Feedback matrix
B	=	Policy matrix
n	=	Time
T	=	Time horizon
k_1, k_2	=	Feedback gains
Ω	=	The set of all admissible policies

$\Phi(L)$ = $k \times k$ matrix polynomial in the lag operator L

V = Variance-covariance matrix

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